SKETCH OF SOLUTIONS (HOMEWORK II)

Chapter 1, 5-9 Think of the polygon with n sides. The following geometric facts from isometries of the polygon are useful:

(1) Any isometry must map an edge to another edge (maybe reversing orientation).
(2) Any isometry is determied by the action on one oriented edge.
(3) There are $2n$ oriented edges

Using these facts we get that the order of D_n is $2n$. Also notice that a reflection reverses the orientation of the edges, while rotating preserves the orientation. The following facts can be verified:

(1) The composition of two rotations is a rotation, therefore it is orientation preserving.
(2) Reflecting twice (about any two axes) gives a mapping which preserves orientation.
(3) Rotating (preserves orientation) and then reflectig reverses orientation.
(4) Reflecting reverses orientation and then rotating preserves orientation.

10. $r_1 r_2 f_1 r_3 f_2 f_3 r_3$ is a reflection (the number of reflections is odd).

Chapter 2

3. $2 \cdot 2 \equiv 0 \mod 4$ and $0 \notin \{1,2,3\}$ therefore $\{1,2,3\}$ is not a group under multiplication $\mod 4$. For $\{1,2,3,4\}$ we have the following multiplication table

	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Using the table it can be verified that multiplication is associative and that there are inverses (1 appears on every row and on every column symmetricaly with respect to the diagonal).

4. Define
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$
Then
$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

5. The inverse of
$$\begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix} \mod 11$$
is
$$\begin{bmatrix} 9 & 9 \\ 10 & 8 \end{bmatrix} \mod 11$$

14. Let $x, y \in G$. Define $a := x, b := yx, c = xy$ then $ab = x(yx) = (xy)x = ca$ therefore $b = c$ that is $yx = xy$.

15. By induction on n
Base $(ab)^1 = a^1b^1$
Inductive Step Suppose $(ab)^{n-1} = a^{n-1}b^{n-1}$. Then
$$(ab)^n = (ab)^{n-1}(ab) = a^{n-1}b^{n-1}(ab)$$
Now we use that G is abelian and we get:
$$a^{n-1}b^{n-1}(ab) = a^{n-1}b^n a = a^{n-1}ab^n = a^n b^n$$
This is not true for non-abelian groups. Take D_4, then
$$(rf)^2 = rfrf = rr^{-1} = e \neq r^2 = r^2e = r^2f^2$$
17. Suppose $(ab)^{-1} = a^{-1}b^{-1}$ $\forall a, b \in G$. Let $x, y \in G$ then $e = xyy^{-1}x^{-1} = xy(yx)^{-1}$ therefore
$$yx = e(yx) = xy(yx)^{-1}(yx) = xy,$$
i.e. G is abelian.

Simply put, since $(ab)^{-1} = b^{-1}a^{-1}$, condition $(ab)^{-1} = a^{-1}b^{-1}$ is equivalent to $a^{-1}b^{-1} = b^{-1}a^{-1}$ $\forall a, b \in G$. Setting $x = a^{-1}, y = b^{-1}$ we get $xy = yx$ $\forall x, y \in G$.
Suppose now that G is abelian. Then $ab = ba$ so $(ab)^{-1} = (ba)^{-1} = a^{-1}b^{-1}$
19. By induction on n
Base $(a^{-1}ba)^{-1} = a^{-1}ba$
Inductive Step Suppose $(a^{-1}ba)^{n-1} = a^{-1}b^{n-1}a$ then
$$(a^{-1}ba)^n = (a^{-1}ba)^{n-1}(a^{-1}ba) = a^{-1}b^{n-1}aa^{-1}ba = a^{-1}b^{n-1}ba = a^{-1}b^n a$$
20. The inverse is $a_n^{-1} \cdot a_{n-1}^{-1} \cdots a_1^{-1}$
25. Using the following facts it is easier to complete the table:
    (1) On a given row (or column) every element of the group must appear listed once (*why?*) and only once (*why?*)
    (2) The identity e must appear symmetrically with respect to the diagonal of the table (*why?*)

The complete table is the following:

	e	a	b	c	d
e	e	a	b	c	d
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c

(**Note:** *Later it will be proved that there is "essentially" only one group of order p for p prime*)
26. Suppose $(ab)^2 = a^2b^2$. Then
$$abab = aabb \therefore aba = aab \therefore ba = ab$$
Suppose now $ba = ab$. Then
$$ba = ab \therefore aba = aab \therefore abab = aabb$$
32. a)
$$fr^{-2}fr^5 = r^2r^5 = r^2r = r^3$$

b)
$$r^{-3}fr^4fr^{-2} = r^2fr^4fr^3 = r^2r^{-4}r^3 = r$$
c)
$$fr^5fr^{-2}f = fr^{-1}fr^{-2}f = rr^{-2}f = r^{-1}f = r^5f$$

34. Multiplication of matrices is associative and the identity matrix is an element of the given set. Therefore the only property that we must show is the existence of inverses on the given set:
$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$