MAT 303 Calculus IV Fall 2003
Midterm II — Take Home Exam

Name:____

I.D.:____

1	2	3	4	5	Extra Credit	TOTAL
20 points	20 points	20 points	20 points	20 points	10 points	100 points

You can use the textbook, lecture notes, and the homework. Show your work, write neatly, and box the final answer.

1. (a) (5 points) Show that the one-parameter family of the straight lines

$$y(x) = Cx + g(C)$$

satisfies the differential equation

$$xy' + g(y') = y.$$

(b) (5 points) Suppose that a body moves through a resisting medium with resistance proportional to its velocity v,

$$\frac{dv}{dt} = -kv, \quad k > 0.$$

Given that $v(0) = v_0$ and $x(0) = x_0$, find the velocity $v(t)$ and the displacement $x(t)$.

(c) (10 points) Suppose that a body is dropped ($v_0 = 0$) from a distance $r_0 > R$ from the earth's center (R is the radius of the earth), so that its acceleration is

$$\frac{dv}{dt} = -\frac{GM}{r^2}.$$

Find the time when a body reaches the height $r < r_0$. *Hint*: Use that $dv/dt = v(dv/dr)$ and use the substitution $r = r_0 \cos^2\theta$ to evaluate the integral $\int \sqrt{r/(r_0 - r)}dr$.

2. (20 points) Using the substitution $v = \ln x$ for the independent variable $x > 0$, transform the differential equation

$$x^2y'' - 6xy' + 6y = 0 \qquad (1)$$

into the constant coefficient linear differential equation with the independent variable v. Using this method, find the general solution of the differential equation (1).

3. (a) (10 points) Find the general solution of the differential equation

$$y^{(4)} + y^{(3)} - y'' + y' - 2y = 0.$$

(b) (10 points) For the differential equation in part (a), solve the initial value problem

$$y(0) = y'(0) = y''(0) = 0,\ y'''(0) = 30.$$

4. Consider the mass-spring-dashpot system with the displacement $x(t)$

(2) $$\ddot{x} + 2p\dot{x} + \omega_0^2 x = 0,$$

where $p = c/(2m)$ and $\omega_0^2 = k/m$, and suppose that the system is underdamped.

(a) (10 points) For the differential equation (2), solve the initial value problem

$$x(0) = x_0,\ v(0) = v_0.$$

(b) (5 points) Write the solution $x(t)$ to the part (b) in the form

$$x(t) = Ce^{-pt}cos(\omega_1 t - \alpha),$$

where $\omega_1^2 = \omega_0^2 - p^2$, and determine C and α.

(c) (5 points) Using part (c), prove that local maxima and minima of the solution $x(t)$ occur when

$$\tan(\omega_1 t - \alpha) = -\frac{p}{\omega_1},$$

and find the difference between two consecutive local maxima of $x(t)$.

5. (a) (10 points) Find a particular solution y_p of the differential equation

$$y''' + y' = 2 - \cos x.$$

(b) (5 points) Set up the appropriate form of a particular solution y_p of the differential equation

$$D^2(D-1)^3(D+2)^2(D^2+4)y = e^{-x} + e^{3x}\cos x.$$

Do not determine the values of the coefficients!

(c) (5 points) Set up the appropriate form of a particular solution y_p of the differential equation

$$D^2(D-1)^3(D+2)^2(D^2+4)y = x^2 + e^x + e^{3x}\cos 2x.$$

Do not determine the values of the coefficients!

Extra Credit Consider the second order linear differential equation with constant coefficients

$$ay'' + by' + cy = 0. \tag{3}$$

The independent variable x is missing, so that it can be solved by the method described in the end of Section 1.6.

(a) (5 points) Show that this method reduces differential equation (3) to the first order homogeneous differential equation with respect to the variable y, and write down a first order separable differential equation corresponding to the homogeneous equation.

(b) (5 points) Suppose that the roots r_1, r_2 of the characteristic equation

$$ar^2 + br + c = 0$$

are distinct. Show that equilibrium solutions of the separable equation in part (a) correspond to the solutions $y_1(x) = e^{r_1 x}$ and $y_2(x) = e^{r_2 x}$ of the differential equation (3).