MAT 303 Calculus IV Fall 2003
Midterm II — Solutions

1. (a) (5 points) Show that the one-parameter family of the straight lines

$$y(x) = Cx + g(C)$$

satisfies the differential equation

$$xy' + g(y') = y.$$

Solution. Trivial: $y' = C$ so that $xy' + g(y') = xC + g(C) = y$.

(b) (5 points) Suppose that a body moves through a resisting medium with resistance proportional to its velocity v,

$$\frac{dv}{dt} = -kv, \quad k > 0.$$

Given that $v(0) = v_0$ and $x(0) = x_0$, find the velocity $v(t)$ and the displacement $x(t)$.

Solution. Clearly, $\boxed{v(t) = v_0 e^{-kt}}$ and $x(t) = C - \frac{v_0}{k} e^{-kt}$. From $x(0) = x_0$ we get $C = x_0 + \frac{v_0}{k}$, so that

$$\boxed{x(t) = x_0 + \frac{v_0}{k}\left(1 - e^{-kt}\right)}$$

(c) (10 points) Suppose that a body is dropped ($v_0 = 0$) from a distance $r_0 > R$ from the earth's center (R is the radius of the earth), so that its acceleration is

$$\frac{dv}{dt} = -\frac{GM}{r^2}.$$

Find the time when a body reaches the height $r < r_0$. *Hint*: Use that $dv/dt = v(dv/dr)$ and use the substitution $r = r_0 \cos^2 \theta$ to evaluate the integral $\int \sqrt{r/(r_0 - r)}dr$.

Solution. We have $v = dr/dt$, where r is the distance from the earth's center. Since

$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr},$$

(as it was suggested in the hint) we get from Newton's law of gravitation

$$v\frac{dv}{dr} = -\frac{GM}{r^2}.$$

Integrating this from r to r_0 and using $v_0 = 0$ we get

$$\frac{v^2}{2} = \frac{GM}{r} - \frac{GM}{r_0}$$

(this is the conservation of energy: kinetic energy + potential energy = constant). Since the body is falling, v is negative, and we get, after some simple algebra,

$$v = \frac{dr}{dt} = -\sqrt{2GM}\sqrt{\frac{r_0 - r}{r_0 r}}.$$

From this equation we get

$$\frac{dt}{dr} = -\sqrt{\frac{r_0}{2GM}}\sqrt{\frac{r}{r_0 - r}}$$

Integrating from r to r_0 we obtain

$$t = \sqrt{\frac{r_0}{2GM}}\int_r^{r_0}\sqrt{\frac{r}{r_0 - r}}\,dr.$$

To compute the integral, we use the substitution $r = r_0\cos^2\theta$, as suggested. Since

$$dr = -2r_0\cos\theta\sin\theta d\theta \quad \text{and} \quad \sqrt{\frac{r}{r_0 - r}} = \frac{\cos\theta}{\sin\theta},$$

$$\int\sqrt{\frac{r}{r_0 - r}}\,dr = -r_0\int 2\cos^2\theta\,d\theta$$
$$= -r_0(\theta + \tfrac{1}{2}\sin 2\theta) = -r_0(\theta + \sin\theta\cos\theta).$$

Finally, using $\theta = \cos^{-1}\sqrt{r/r_0}$, we get after simple algebra,

$$\boxed{t = \sqrt{\frac{r_0}{2GM}}\left(\cos^{-1}\sqrt{\frac{r}{r_0}} + \sqrt{rr_0 - r^2}\right)}$$

(which is the answer to the problem 28(a) in Section 2.3).

2. (20 points) Using the substitution $v = \ln x$ for the independent variable $x > 0$, transform the differential equation

$$x^2y'' - 6xy' + 6y = 0 \tag{1}$$

into the constant coefficient linear differential equation with the independent variable v. Using this method, find the general solution of the differential equation (**??**).

Solution. We get but the chain rule,

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = \frac{1}{x}\frac{dy}{dv}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dv}\right) = -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dv}\right)$$
$$= -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x}\frac{d^2y}{dv^2}\frac{dv}{dx} = -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x^2}\frac{d^2y}{dv^2}.$$

Substituting these expressions into the differential equation (**??**), we get

$$x^2y'' - 6xy' + 6y = \frac{d^2y}{dv^2} - 7\frac{dy}{dv} + 6y = 0,$$

where now v is an independent variable. The characteristic equation is

$$r^2 - 7r + 6 = 0$$

and its roots are $r_1 = 1, r_2 = 6$. Thus

$$y(v) = c_1e^v + c_2e^{6v}$$

and

$$\boxed{y(x) = c_1x + c_2x^6}$$

3. (a) (10 points) Find the general solution of the differential equation

$$y^{(4)} + y^{(3)} - y'' + y' - 2y = 0.$$

Solution. The characteristic equation is

$$r^4 + r^3 - r^2 + r - 2 = 0$$

and, by inspection, has a root $r_1 = 1$. Factoring, we rewrite the characteristic equation as

$$(r-1)(r^3 + 2r^2 + r + 2) = (r-1)(r+2)(r^2+1) = 0.$$

The roots are $1, -2, i, -i$ and the general solution is

$$\boxed{y(x) = c_1e^x + c_2e^{-2x} + c_3\cos x + c_4\sin x}$$

(b) (10 points) For the differential equation in part (a), solve the initial value problem

$$y(0) = y'(0) = y''(0) = 0,\ y'''(0) = 30.$$

Solution. We have the following system of linear algebraic equations for the constants c_1, c_2, c_3 and c_4:

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_1 - 2c_2 + c_4 &= 0 \\ c_1 + 4c_2 - c_3 &= 0 \\ c_1 - 8c_2 - c_4 &= 30 \end{aligned}$$

Solving this system, we get $c_1 = 5, c_2 = -2, c_3 = -3, c_4 = -9$, so that

$$\boxed{y(x) = 5e^x - 2e^{-2} - 3\cos x - 9\sin x}$$

4. Consider the mass-spring-dashpot system with the displacement $x(t)$

(2) $$\ddot{x} + 2p\dot{x} + \omega_0^2 x = 0,$$

where $p = c/(2m)$ and $\omega_0^2 = k/m$, and suppose that the system is underdamped.

(a) (10 points) For the differential equation (2), solve the initial value problem

$$x(0) = x_0,\ v(0) = v_0.$$

Solution. General solution is $x(t) = e^{-pt}(A\cos\omega_1 t + B\sin\omega_1 t)$. Using $x(0) = x_0$ we get $A = x_0$ and

$$\dot{x}(0) = -pA + B\omega_1 = v_0, \quad B = \frac{v_0 + px_0}{\omega_1}.$$

Thus

$$\boxed{x(t) = e^{-pt}\left(x_0\cos\omega_1 t + \frac{v_0 + px_0}{\omega_1}\sin\omega_1 t\right)}$$

(b) (5 points) Write the solution $x(t)$ to the part (b) in the form

$$x(t) = Ce^{-pt}cos(\omega_1 t - \alpha),$$

where $\omega_1^2 = \omega_0^2 - p^2$, and determine C and α.

Solution.

$$\boxed{C = \sqrt{x_0^2 + \left(\frac{v_0 + px_0}{\omega_1}\right)^2}}$$

and

$$\boxed{\cos\alpha = \frac{x_0}{C},\ \sin\alpha = \frac{v_0 + px_0}{\omega_1 C}}$$

(c) (5 points) Using part (b), prove that local maxima and minima of the solution $x(t)$ occur when

$$\tan(\omega_1 t - \alpha) = -\frac{p}{\omega_1},$$

and find the difference between two consecutive local maxima of $x(t)$.

Solution. Critical points are solutions of $\dot{x}(t) = 0$ and from part (b),

$$\dot{x}(t) = -pCe^{-pt}\cos(\omega_1 t - \alpha) - Ce^{-pt}\omega_1 \sin(\omega_1 t - \alpha) = 0,$$

so

$$\boxed{\tan(\omega_1 t - \alpha) = -\frac{p}{\omega_1}}$$

Either from the graph in the textbook (Figure 3.4.9), or using second derivative test, we see that critical points alternate between local maxima and minima, so the difference between two consecutive local maxima is

$$\boxed{\frac{2\pi}{\omega_1}}$$

(compare with problems 30,32 in Section 3.4).

5. (a) (10 points) Find a particular solution y_p of the differential equation

$$y''' + y' = 2 - \cos x.$$

Solution. Characteristic equation is $r^3 + r = 0$, so the roots are $0, i, -i$ and complementary solution is

$$y_c(x) = c_1 + c_2 \cos x + c_3 \sin x.$$

In this case, both 2 and $-\cos x$ appear in y_c, so the Rule 2 applies and the trial form of a particular solution is

$$y_p(x) = Ax + x(B\cos x + C\sin x).$$

Computing y_p', y_p'', y_p''' and substituting them into the differential equation, we get

$$y_p''' + y_p' = A - 2B\cos x - 2C\sin x = 2 - \cos x.$$

Thus $A = 2, B = 1/2, C = 0$ and

$$\boxed{y_p(x) = 2 + \tfrac{1}{2}x\cos x}$$

(b) (5 points) Set up the appropriate form of a particular solution y_p of the differential equation

$$D^2(D-1)^3(D+2)^2(D^2+4)y = e^{-x} + e^{3x}\cos x.$$

Do not determine the values of the coefficients!
Solution. Roots of the characteristic equation are

$$-2, -2, 0, 0, 1, 1, 1, 2i, -2i$$

and we see that neither $f(x) = e^{-x} + e^{3x}\cos x$, nor its derivatives appear in the complementary solution. Thus the Rule 1 applies and the trial form of a particular solution is

$$\boxed{y_p(x) = Ae^{-x} + e^{3x}(B\cos x + C\sin x)}$$

(c) (5 points) Set up the appropriate form of a particular solution y_p of the differential equation

$$D^2(D-1)^3(D+2)^2(D^2+4)y = x^2 + e^x + e^{3x}\cos 2x.$$

Do not determine the values of the coefficients!
Solution. The roots are the same as in part (b), but now derivative of the first term in $f(x)$ and the second term appear in a complementary solution. The latter is

$$y_c(x) = c_1 + c_2x + e^{-2x}(c_3 + c_4x) + e^x(c_5 + c_6x + c_7x^2) + c_8\cos 2x + c_9\sin x.$$

Thus, according to Rule 2, the trial form of a particular solution is

$$\boxed{y_p(x) = x^2(Ax^2 + Bx + C) + Dx^3e^x + e^{3x}(E\cos 2x + F\sin 2x)}$$

Extra Credit Consider the second order linear differential equation with constant coefficients

$$ay'' + by' + cy = 0. \tag{3}$$

The independent variable x is missing, so that it can be solved by the method described in the end of Section 1.6.

(a) (5 points) Show that this method reduces differential equation (3) to the first order homogeneous differential equation with respect to the variable y, and write down a first order separable differential equation corresponding to the homogeneous equation.

Solution. Using the substitution $p = y'$ and treating p as a function of y, we get $y'' = p\,dp/dy$ and the differential equation (3) becomes

$$\frac{dp}{dy} = -\frac{b}{a} - \frac{y}{p}\frac{c}{a}$$

(substituting and dividing by ap). This is a homogeneous first order differential equation, and the substitution $v(y) = p(y)/y$ reduces it to the separable differential equation

$$\boxed{y\frac{dv}{dy} = -\frac{av^2 + bv + c}{av}}$$

(b) (5 points) Suppose that the roots r_1, r_2 of the characteristic equation

$$ar^2 + br + c = 0$$

are distinct. Show that equilibrium solutions of the separable equation in part (a) correspond to the solutions $y_1(x) = e^{r_1 x}$ and $y_2(x) = e^{r_2 x}$ of the differential equation (3).

Solution. The equilibrium solutions of the separable equation in part (a) are roots of the equation $av^2 + bv + c = 0$, i.e., $v = r_1$ and $v = r_2$. But if $v = r$, then $p = ry$ or, equivalently, $y' = ry$, which gives $y(x) = Ce^{rx}$, as wanted.