MAT 303 Calculus IV Fall 2003
Midterm III — Solutions

1. For differential equations below set up the appropriate form of a particular solution y_p. DO NOT determine the values of the coefficients.

(a) (10 points)

$$(D-2)^4(D^2-9)^2 y = xe^{2x} + e^{3x} + e^{-x}.$$

Solution.
Roots of the characteristic equations are: $2, 2, 2, 2, 3, 3, -3, -3$, and the complementary solution is

$$y_c(x) = (c_1 + c_2 x + c_3 x^2 + c_4 x^3)e^{2x} + (c_5 + c_6 x)e^{3x} + (c_7 + c_8 x)e^{-3x},$$

so that the appropriate form of a particular solution is

$$\boxed{y_p(x) = x^4(Ax+B)e^{2x} + Cx^2e^{3x} + De^{-x}}$$

(b) (10 points)

$$y^{(4)} + 8y'' + 16y = x^2 \cos 2x.$$

Solution
The characteristic equation is

$$r^4 + 8r^2 + 16 = 0 \quad \text{or} \quad (r^2+4)^2 = 0,$$

so the roots are $2i, 2i, -2i, -2i$ and the complementary solution is

$$y_c(x) = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x.$$

Thus the appropriate form of a particular solution is

$$\boxed{y_p(x) = x^2\left\{(Ax^2+Bx+C)\cos 2x + (Dx^2+Ex+F)\sin 2x\right\}}$$

2. (20 points) Use the method of variation of parameters to find a particular solution y_p of the differential equation

$$y'' + y = \sin x.$$

Hint: You may use the formula $\int \sin^2 x\,dx = \frac{1}{2}(x - \sin x \cos x)$.
Solution.

The roots of the characteristic equation are $i, -i$ and the complementary solution is

$$y_c(x) = c_1 \cos x + c_2 \sin x.$$

According to the method, we set

$$y_p(x) = u_1(x)\cos x + u_1(x)\sin x,$$

where the unknown functions u_1 and u_2 satisfy

$$u_1' \cos x + u_2' \sin x = 0.$$

We have, using this condition,

$$\begin{aligned} y_p' &= -\, u_1 \sin x + u_2 \cos x, \\ y_p'' &= -\, u_1' \sin x + u_2' \cos x - u_1 \cos x - u_2 \sin x. \end{aligned}$$

Substituting this into the differential equation $y'' + y = \sin x$, we get

$$-u_1' \sin x + u_2' \cos x = \sin x.$$

Together with the condition above, this gives a system of two linear algebraic equations for the unknown functions u_1' and u_2':

$$\begin{aligned} u_1' \cos x + u_2' \sin x &= 0, \\ -u_1' \sin x + u_2' \cos x &= \sin x. \end{aligned}$$

Using the basic identity $\cos^2 x + \sin^2 x = 1$, we easily get

$$u_1' = -\sin^2 x, \;\; u_2' = \cos x \sin x.$$

Integrating and using the hint for determining u_1 and the substitution $u = \sin x$, $du = \cos x\, dx$ for determining u_2, we obtain

$$u_1(x) = \tfrac{1}{2}(\sin x \cos x - x), \;\; u_2(x) = \tfrac{1}{2} \sin^2 x.$$

Plugging it back into the formula for y_p and using the basic identity once again, we finally get

$$\boxed{y_p(x) = \tfrac{1}{2} \sin x - \tfrac{1}{2} x \cos x}$$

Note that the first term is a homogeneous solution, so we could also take $y_p(x) = -\frac{1}{2} x \cos x$ as a particular solution.

3. (20 points) Find a general solution of the following system

$$\begin{array}{rcrrr} x_1' &=& 5x_1 & & -6x_3 \\ x_2' &=& 2x_1 & -x_2 & -2x_3 \\ x_3' &=& 4x_1 & -2x_2 & -4x_3 \end{array}$$

Solution.

Set

$$\mathbf{A} = \begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}.$$

Expanding with respect to the first row and simplifying, we easily get

$$|\mathbf{A} - \lambda \mathbf{I}| = -\lambda^3 + \lambda = 0,$$

so that the eigenvalues of the matrix $\mathbf{A}$ are $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = -1$.

The equation $\mathbf{A}\mathbf{v}_1 = \mathbf{0}$ reduces to

$$\begin{aligned} 5v_1 - 6v_3 &= 0 \\ 2v_1 - v_2 - 2v_3 &= 0 \end{aligned}$$

and setting $v_1 = 6$ we find $v_3 = 5$ and $v_2 = 2$, so that

$$\mathbf{v}_1 = \begin{bmatrix} 6 \\ 2 \\ 5 \end{bmatrix}.$$

The equation $(\mathbf{A} - \mathbf{I})\,\mathbf{v}_2 = \mathbf{0}$ reduces to

$$\begin{aligned} 4v_1 - 6v_3 &= 0 \\ 2v_1 - 2v_2 - 2v_3 &= 0 \end{aligned}$$

(the third equation follows from these two equations) and setting $v_1 = 3$ we find $v_3 = 2$ and $v_2 = 1$, so that

$$\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

The equation $(\mathbf{A} + \mathbf{I})\,\mathbf{v}_3 = \mathbf{0}$ reduces to

$$\begin{aligned} 6v_1 - 6v_3 &= 0 \\ 4v_1 - 2v_2 - 3v_3 &= 0 \end{aligned}$$

and setting $v_1 = 2$ we find $v_3 = 2$ and $v_2 = 1$, so that

$$\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

Putting everything together,

$$\mathbf{x}(t) = c_1\mathbf{v}_1 + c_2e^t\mathbf{v}_2 + c_3e^{-t}\mathbf{v}_3 = \begin{bmatrix} 6c_1 + 3c_2e^t + 2c_3e^{-t} \\ 2c_1 + c_2e^t + c_3e^{-t} \\ 5c_1 + 2c_2e^t + 2c_3e^{-t} \end{bmatrix}.$$

4. (20 points) Find the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix}.$$

Solution. Since $\mathbf{A}$ is upper-triangular matrix,

$$|\mathbf{A} - \lambda\mathbf{I}| = (\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4)$$

so the eigenvalues are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3, \lambda_4 = 4$.

The equation $(\mathbf{A} - \mathbf{I})\,\mathbf{v}_1 = \mathbf{0}$ immediately gives $v_1 = 1$ and, subsequently, $v_2 = -2$, $v_3 = 3$ and $v_4 = -4$, so that

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix}.$$

The equation $(\mathbf{A} - 2\mathbf{I})\,\mathbf{v}_2 = \mathbf{0}$ immediately gives $v_1 = 0$ and $v_2 = 1$ so that $v_3 = -3$ and $v_4 = 6$,

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -3 \\ 6 \end{bmatrix}.$$

The equation $(\mathbf{A} - 3\mathbf{I})\,\mathbf{v}_3 = \mathbf{0}$ immediately gives $v_1 = v_2 = 0$ and $v_3 = 1$, $v_4 = -4$,

$$\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -4 \end{bmatrix}.$$

Finally, the equation $(\mathbf{A} - 4\mathbf{I})\,\mathbf{v}_4 = \mathbf{0}$ immediately gives $v_1 = v_2 = v_3 = 0$ and $v_4 = 1$,

$$\mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Putting everything together,

$$\mathbf{x}(t) = c_1e^t\mathbf{v}_1 + c_2e^{2t}\mathbf{v}_2 + c_3e^{3t}\mathbf{v}_3 + c_4e^{4t}\mathbf{v}_4 = \begin{bmatrix} c_1e^t \\ -2c_1e^t + c_2e^{2t} \\ 3c_1e^t - 3c_2e^{2t} + c_3e^{3t} \\ -4c_1e^t + 6c_2e^{2t} - 4c_3e^{3t} + c_4e^{4t} \end{bmatrix}.$$

5. (20 points) Find the general solution of the following system

$$\begin{aligned} x_1' &= 5x_1 - 9x_2 \\ x_2' &= 2x_1 - x_2 \end{aligned}$$

Solution.

Set

$$\mathbf{A} = \begin{bmatrix} 5 & -9 \\ 2 & -1 \end{bmatrix}.$$

We compute

$$|\mathbf{A} - \lambda \mathbf{I}| = \lambda^2 - 4\lambda + 13 = (\lambda - 2)^2 + 9 = 0,$$

so that the eigenvalues are $2 + 3i, 2 - 3i$.

The equation $(\mathbf{A} - (2+3i)\mathbf{I})\,\mathbf{v}_1 = \mathbf{0}$ gives

$$(3 - 3i)v_1 - 9v_2 = 0$$

(the second equation is equivalent to this one) and we could set $v_1 = 3$ and $v_2 = 1 - i$ (any other non-zero choice of v_1 is also fine). Thus the complex-valued solution is $\mathbf{x}_1(t) = e^{(2+3i)t}\mathbf{v}_1$, and separating its real and imaginary parts,

$$\begin{aligned}
\mathbf{x}_1(t) &= e^{2t}(\cos 3t + i \sin 3t) \left(\begin{bmatrix} 3 \\ 1 \end{bmatrix} - i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\
&= e^{2t} \left(\cos 3t \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \sin 3t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + i e^{2t} \left(\sin 3t \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \cos 3t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\
&= e^{2t} \begin{bmatrix} 3\cos 3t \\ \cos 3t + \sin 3t \end{bmatrix} + i e^{2t} \begin{bmatrix} 3 \sin 3t \\ \sin 3t - \cos 3t \end{bmatrix}.
\end{aligned}$$

Taking these real and imaginary parts as linear independent real-valued solutions, we finally get

$$\mathbf{x}(t) = \begin{bmatrix} e^{2t}(3c_1 \cos 3t + 3c_2 \sin 3t) \\ e^{2t}\left(c_1(\cos 3t + \sin 3t) + c_2(\sin 3t - \cos 3t)\right). \end{bmatrix}$$