MAT 303 Calculus IV Spring 2004
Midterm II — Solutions

1. (a) (5 points) Find the critical points of the autonomous differential equation

$$\frac{dx}{dt} = x(4 - x^2).$$

Solution. $f(x) = x(4 - x^2) = x(2 - x)(2 + x)$ and the critical points are the roots of the equation $f(x) = 0$, i.e., $-2, 0, 2$.

(b) (5 points) Construct the phase diagram for the differential equation in part (a) and for each critical point determine whether it is stable or unstable.

Solution. The phase diagram is

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-------->-2<----------0---------->2<----------
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so that the equilibrium solution $x = 0$ is unstable and solutions $x = \pm 2$ are stable.

2. (a) (10 points) Verify that $y_1 = x^2$ and $y_2 = x^3$ are two different solutions of the differential equation

$$x^2 y'' - 4xy' + 6y = 0,$$

satisfying $y(0) = y'(0) = 0$. Explain why these facts do not contradict the existence and uniqueness theorem for second order linear differential equations.

Solution. The verification is straightforward. To apply the theorem, we need to rewrite the differential equation in the normalized form

$$y'' + p(x)y' + q(x)y = 0.$$

Dividing our equation by x^2, we get that

$$p(x) = -\frac{4}{x}, \ q(x) = \frac{6}{x^2}.$$

The coefficients $p(x)$ and $q(x)$ are not continuous at $x = 0$, so that the existence and uniqueness theorem does not apply to the IVP $y(0) = y'(0) = 0$. Thus the fact that this IVP has two different solutions does not contradict the existence and uniqueness theorem.

(b) (10 points) Solve the following initial value problem

$$y'' + 2y' + y = 0, \quad y(0) = 1, \ y'(0) = 0.$$

Solution. The characteristic equation is

$$r^2 + 2r + 1 = (r + 1)^2 = 0$$

and has a double root $r = -1$. Thus a general solution is

$$y = (c_1 + c_2 x)e^{-x}.$$

Since $y' = -(c_1 + c_2 x)e^{-x} + c_2 e^{-x}$, plugging in $x = 0$ we get $c_1 = 1$ and $c_2 - c_1 = 0$, i.e., $c_2 = 1$ as well. Thus

$$\boxed{y = (1 + x)e^{-x}}$$

3. (a) (15 points) Find a general solution of the differential equation

$$y''' - 6y'' + 25y' = 0.$$

Solution. The characteristic equation is

$$r^3 - 6r^2 + 25r = r(r^2 - 6r + 25) = 0$$

and its roots are $r_1 = 0$ and $r_{2,3} = 3 \pm 4i$. Thus a general solution is

$$\boxed{y = c_1 + e^{3x}(c_2 \cos 4x + c_3 \sin 4x)}$$

(b) (15 points) Determine an appropriate form for a particular solution of the following differential equation

$$D^2(D-1)^2(D^2+9)(D-5)^3 y = x^2 + e^{-x} + \cos 3x + xe^{5x}.$$

DO NOT DETERMINE THE VALUES OF THE COEFFICIENTS!
Solution. The roots of the characteristic equation are

$$0, 0, 1, 1, 3i, -3i, 5, 5, 5$$

so that the complementary solution is

$$y_c(x) = (c_1 + c_2 x) + e^x(c_3 + c_4 x) + c_5 \cos 3x + c_6 \sin 3x + e^{5x}(c_7 + c_8 x + c_9 x^2).$$

Using Rules 1 and 2, we find that an appropriate form for a particular solution is

$$\boxed{y_p(x) = x^2(Ax^2 + Bx + C) + De^{-x} + x(E\cos 3x + F \sin 3x) + x^3(Hx + I)e^{5x}}$$

4. (20 points) Solve the following initial value problem:

$$x'' + 16x = -8 \sin 4t, \quad x(0) = 0, \ x'(0) = 0.$$

Solution. The easiest way is to use the method of undetermined coefficients (you can also use the variation of parameters method, but it will involve more computations). The characteristic equation is

$$r^2 + 16 = 0, \quad r_{1,2} = \pm 4i,$$

and

$$x_c(t) = c_1 \cos 4t + c_2 \sin 4t.$$

Since the nonhomogeneous term appears in the complementary solution, we need to use Rule 2. Thus

$$x_p(t) = t(A \cos 4t + B \sin 4t).$$

We find that

$$x' = A \cos 4t + B \sin 4t - 4At \sin 4t + 4Bt \cos 4t$$

and

$$x'' = -8A \sin 4t + 8B \cos 4t - 16At \cos 4t - 16Bt \sin 4t.$$

Substituting this into the differential equation, we get

$$x'' + 16x = -8A \sin 4t + 8B \cos 4t = -8 \sin 4t,$$

i.e., $A = 1, B = 0$ and

$$x_p(t) = t \cos 4t.$$

Thus a general solution is

$$x(t) = x_c(t) + x_p(t) = c_1 \cos 4t + c_2 \sin 4t + t \cos 4t.$$

Substituting $t = 0$ into $x(t)$ and $x'(t)$, we find that $c_1 = 0$ and $c_2 = -\frac{1}{4}$, i.e.,

$$\boxed{x(t) = -\frac{1}{4} \sin 4t + t \cos 4t}$$

5. (20 points) It is given that all eigenvalues of the endpoint problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \; y'(L) = 0$$

on the interval $[0, L]$ are non-negative. Find all eigenvalues and associated eigenfunctions for this problem.

Solution. The characteristic equation is $r^2 + \lambda = 0$; since λ is non-negative, $\lambda = a^2$ for some real a, and

$$r_{1,2} = \pm ai.$$

A general solution of our differential equation is

$$y(x) = A \cos ax + B \sin ax$$

and

$$y'(x) = -aA \sin ax + aB \cos ax.$$

At $x = 0$ we have $y'(0) = aB = 0$, so that either $a = 0$ and B is arbitrary, or $a \neq 0$ and $B = 0$. In the former case, $\lambda = 0$ is the eigenvalue and corresponding eigenfunction is $y(x) = \text{const}$.

In the latter case, $y(x) = A\cos ax$ and $A \neq 0$. The second endpoint condition $y'(L) = 0$ reads $aA\sin aL = 0$, and since both a and A are non-zero, we should have $\sin aL = 0$, so that

$$aL = \pi n,\ a = \frac{\pi n}{L}, \quad n = \pm 1, \pm 2, \ldots$$

The corresponding eigenvalues are $\lambda_n = \left(\frac{\pi n}{L}\right)^2$ and corresponding eigenfunctions are $y_n(x) = \cos\frac{\pi n x}{L}$. Since cos is an even function, we see that n and $-n$ give the same eigenvalue and eigenfunction; moreover, the eigenvalue $\lambda = 0$ with the eigenfunction $y = \text{const}$ corresponds to $n = 0$. Thus

$$\boxed{\lambda_n = \left(\frac{\pi n}{L}\right)^2, \quad y_n(x) = \cos\frac{\pi n x}{L}, \quad n = 0, 1, 2, \ldots}$$

Extra Credit Let M be the mass of the Earth, R — the radius of the Earth and G — the gravitational constant. Suppose that a rocket is fired from the surface of the Earth with initial velocity $v_0 > 0$.

(a) (7 points) Using Newton's law of gravitation, derive the differential equation for $v = dr/dt$ as a function of $r(t)$ — the rocket's distance from the center of the Earth at time t.

Solution. By Newton's laws,

$$ma = m\frac{dv}{dt} = -G\frac{mM}{r^2},$$

i.e.,

$$\frac{dv}{dt} = -\frac{GM}{r^2}.$$

By the chain rule,

$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr}.$$

Thus from the Newton's laws we get the following differential equation for the function $v(r)$:

$$\boxed{v\frac{dv}{dr} = -\frac{GM}{r^2}}$$

(b) (8 points) Using the differential equation derived in part (a), find the minimal initial velocity v_0 such that the rocket's velocity $v(t)$ will be positive at all times t, i.e., the rocket continues forever to move away from the Earth.

Solution. The equation derived in part (a) is separable, so that

$$\int v dv = -GM \int \frac{dr}{r^2},$$

or

$$\frac{v^2}{2} = \frac{GM}{r} + C.$$

At $t = 0$ we have that $r = R$ and $v(R) = v_0$, so that

$$C = \frac{v_0^2}{2} - \frac{GM}{R}$$

and

$$\frac{v^2}{2} = \frac{GM}{r} + \frac{v_0^2}{2} - \frac{GM}{R}$$

The velocity $v(r)$ will be positive for all times provided

$$\frac{v_0^2}{2} - \frac{GM}{R} \geq 0$$

i.e.

$$\boxed{v_0 \geq \sqrt{\frac{2GM}{R}}}$$