MAT 303 Calculus IV Spring 2004
Midterm I — Solutions

1. (a) (5 points) Find general solution of the differential equation

$$\frac{dy}{dx} = 2\sqrt{y}$$

Solution. It is a separable equation. Assuming that $y \neq 0$, we get

$$\int \frac{dy}{2\sqrt{y}} = \int dx,$$

or

$$\sqrt{y} = x + C.$$

Since always $\sqrt{y} \geq 0$, this solution exists only when $x+C \geq 0$, i.e., when $\geq -C$. Thus

$$\boxed{y = (x+C)^2 \quad \text{for} \quad x \geq -C}$$

is a general solution and $\boxed{y = 0}$ is the singular solution.

(b) (5 points) For the differential equation in part (a) find the solution of the initial value problem $y(0) = 1$. Check whether the existence and uniqueness Theorem (Theorem I) applies to this case.

Solution. The differential equation has the form $\frac{dy}{dx} = f(x,y)$ with $f(x,y) = 2\sqrt{y}$. This function, as well as $\frac{\partial f}{\partial y} = \frac{1}{\sqrt{y}}$ are both obviously continuous around the point $(0,1)$ (that is, in some neighborhood of the point $(0,1)$ on xy-plane). Thus $\boxed{\text{Theorem I applies}}$ to this initial condition, and the IVP has a unique solution on some interval containing the point $x = 0$. Plugging the initial condition into the general solution, we get $y(0) = C^2 = 1$, so that $C = \pm 1$. However, for $C = -1$ at $x = 0$ we have $x - 1 = -1 < 0$, so that by part (a) this case is out. Thus the solution is $y(x) = (x+1)^2$. Again by part (a), this solution is defined only when $x + 1 \geq 0$, i.e., when $x \geq -1$. Since $y(-1) = 0$ and $y = 0$ is a solution, we can extend the function $y(x)$ to $x < -1$ by setting it to be 0. Thus the solution of the IVP defined for all x is

$$\boxed{y(x) = \begin{cases} (x+1)^2, & x \geq -1, \\ 0, & x < -1. \end{cases}}$$

(c) (5 points) Does Theorem I apply to the initial value problem $y(0) = -1$ for the differential equation in part (a)? How many different solutions (if any) this initial value problem has?

Solution. The function $f(x, y) = 2\sqrt{y}$ is not defined at the point $(0, -1)$, so that $\boxed{\text{Theorem I does not apply}}$ (so that it does proved any information on existence and uniqueness of solutions). In this case, $\boxed{\text{there is no solution}}$ since y can not take negative values (or you can not find real values of C to meet $y(0) = -1$).

(d) (5 points) Does Theorem I apply to the initial value problem $y(0) = 0$ for the differential equation in part (a)? How many different solutions (if any) this initial value problem has?

(*Hint:* You may use the fact the for every constant c the function $y(x)$ defined by

$$y(x) = \begin{cases} 0, & \text{for } x \le c, \\ (x-c)^2, & \text{for } x > c \end{cases}$$

satisfies differential equation in part (a))

Solution. The function $f(x, y) = 2\sqrt{y}$ is not defined in a neighborhood of point $(0, 0)$ (since values $y < 0$ are not allowed) so that $\boxed{\text{Theorem I does not apply.}}$ (Also, we can check that $\frac{\partial f}{\partial y} = \frac{1}{\sqrt{y}}$ blows up at $y = 0$.) However, every function $y(x)$ in the Hint with $c \ge 0$ satisfies the initial condition $y(0) = 0$, so that this IVP $\boxed{\text{has infinitely many solutions.}}$

2. (a) (10 points) Find general solution of the differential equation

$$xy' - y = x$$

Solution 1. It is a linear equation. Assuming $x \neq 0$, we get

$$y' - \frac{1}{x}y = 1,$$

and we get for the integration factor

$$\rho(x) = e^{\int -dx/x} = e^{-\ln x} = \frac{1}{x}.$$

Multiplying both sides of the equation by $\rho(x)$, we get

$$\left[\frac{1}{x}y\right]' = \frac{1}{x},$$

so that $y/x = \ln|x| + C$, and general solution is

$$\boxed{y = x(\ln|x| + C) \quad \text{for} \quad x \neq 0}$$

Solution 2. This is also a homogeneous equation:

$$y' - \frac{y}{x} = 1$$

for $x \neq 0$. Setting $v = y/x$ we have $y = xv$ and $y' = v + xv'$. The differential equation becomes

$$xv' + v - v = 1, \quad \text{or} \quad xv' = 1.$$

Thus

$$v' = \frac{1}{x}, \quad v = \int \frac{dx}{x} = \ln|x| + C$$

and $\boxed{y = x(\ln|x| + C)}$, as before.

Both solutions 1 and 2 are fine.

(b) (10 points) For the differential equation in part (a) find particular solution of the initial value problem $y(1) = 1$ and determine the maximal interval (a, b) on which the solution exists.

Solution. Setting $y(1) = 1$ and using $\ln 1 = 0$, we get $C = 1$. Since $x \neq 0$, the maximal interval containing point $x = 1$ on which solution exists (and is continuously differentiable function) is $\boxed{(0, \infty).}$ The solution of this IVP is

$$\boxed{y(x) = x(\ln x + 1)}$$

(Note that since $\lim_{x \to 0} x \ln x = 0$, the function $y = x(\ln|x| + 1)$ could be extended to a continous function on the whole real line by setting $y(0) = 0$. However, the derivative of this function will become $-\infty$ at $x = 0$).

3. (20 points) Find general solutions of the following differential equations:

(a) $(1 + ye^{xy})dx + (3y + xe^{xy})dy = 0$

Solution. Set $M(x, y) = 1 + ye^{xy}$ and $N(x, y) = 3y + xe^{xy}$. We have

$$\frac{\partial M}{\partial y} = e^{xy} + xye^{xy} \quad \text{and} \quad \frac{\partial N}{\partial x} = e^{xy} + xye^{xy},$$

so that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

and equation is exact. Solving

$$\frac{\partial F}{\partial x} = M(x,y) = 1 + ye^{xy}$$

we get $F(x,y) = x + e^{xy} + g(y)$, and substituting this into

$$\frac{\partial F}{\partial y} = N(x,y) = 3y + xe^{xy}$$

we get $g' = 3y$, so that $g(y) = 3y^2/2$. Thus the general solution is

$$\boxed{F(x,y) = \frac{3}{2}y^2 + x + e^{xy} = C}$$

(b) $x^2y' = xy + y^2$

Solution. Rewriting the differential equation as

$$y' = \frac{y}{x} + \frac{y^2}{x^2}$$

(and assuming $x \neq 0$) we see that it is homogeneous equation. Making the substitution $v = y/x$, we have from $y = vx$:

$$y' = v + xv',$$

so that the differential equation becomes

$$xv' + v = v + v^2, \quad \text{or} \quad xv' = v^2.$$

This is a separable equation,

$$\int \frac{dv}{v^2} = \int \frac{dx}{x},$$

and we get

$$-\frac{1}{v} = \ln|x| + C, \quad \text{or} \quad v = -\frac{1}{\ln|x| + C}.$$

Thus the general solution is

$$\boxed{y = -\frac{x}{\ln|x| + C} \quad \text{for} \quad x \neq 0}$$

4. (20 points) Find general solution of the differential equation

$$x^2y' + 2xy = 5y^3$$

Solution. This is Bernoulli equation with $n = 3$. Making a substitution $v = y^{1-n} = y^{-2}$ we get $v' = -2y^{-3}y'$. Multiplying both sides of the differential equation by $-2y^{-3}y'$, we can rewrite it as a linear differential equation for the variable v:

$$v' - \frac{4}{x}v = -\frac{10}{x^2}.$$

We get for the integrating factor,

$$\rho(x) = e^{\int -4dx/x} = e^{-4\ln x} = \frac{1}{x^4},$$

so that multiplying by $\rho(x)$ we get

$$\left[\frac{1}{x^4}v\right]' = -\frac{10}{x^6},$$

and

$$\frac{v}{x^4} = \frac{2}{x^5} + C, \quad \text{or} \quad v = Cx^4 + \frac{2}{x}.$$

Remembering that $y = v^{-1/2}$, we get for a general solution,

$$\boxed{y = \frac{1}{\sqrt{Cx^4 + 2x^{-1}}}}$$

5. (20 points) Find general solution of the differential equation

$$y'' = (y')^2$$

Solution 1. This is a reducible second order differential equation where independent variable x is missing. Making the substitution

$$p = \frac{dy}{dx},$$

we get by the Chain Rule,

$$y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy},$$

where we are treating y as an independent variable. Thus the second order differential equation for the function $y(x)$ reduces to the following first order differential equation for the function $p(y)$:

$$pp' = p^2, \quad \text{or} \quad p' = p \quad \text{for} \quad p \neq 0.$$

Its general solution is

$$p(y) = Ae^y$$

with an arbitrary constant A (case $p = 0$ corresponds to $A = 0$). Now remembering that $p = \frac{dy}{dx}$ we get a separable equation for the function $y(x)$:

$$\frac{dy}{dx} = Ae^y, \quad \text{or} \int e^{-y} dy = A \int dx.$$

Thus

$$-e^{-y} = Ax - B,$$

for another arbitrary constant B, and we finally get a general solution

$$\boxed{y = -\ln(B - Ax)}$$

(For a given values of A and B $y(x)$ is defined whenever $B - Ax > 0$.)

Solution 2. This is also a reducible second order where variable y is missing. Setting a before

$$p = \frac{dy}{dx}$$

we get $y'' = p' = dp/dx$ and the equation becomes

$$p' = p^2.$$

This is separable equation so that

$$\int \frac{dp}{p^2} = \int dx, \quad \text{or} \quad -\frac{1}{p} = x + C_1.$$

Thus

$$\frac{dy}{dx} = p = -\frac{1}{x + C_1} \quad \text{and} \quad \boxed{y = -\ln|x + C_1| + C_2},$$

which is the same answer as before (express constants C_1 and C_2 through A and B.

Both solutions 1 and 2 are fine.

Extra Credit (10 points) Find the solution of the initial value problem

$$2x + \frac{dy}{dx} = (x^2 + y)^2, \quad y(0) = 1,$$

and determine the maximal interval (a, b) on which the solution exists.

Solution. Make a substitution $v = x^4 + y$, so that $v' = 4x^3 + y'$. Thus for the new dependent variable v the differential equation becomes

$$\frac{dv}{dx} = v^2.$$

This is a separble equation and we get

$$\int \frac{dv}{v^2} = \int dx, \quad \text{or} \quad -\frac{1}{v} = x + C, \quad \text{or} \quad v = -\frac{1}{x+C}.$$

Thus

$$y = v - x^4 = -\frac{1}{x+C} - x^4.$$

Plugging in $y(0) = 1$ we get $C = -1$ so that

$$\boxed{y(x) = \frac{1}{1-x} - x^4}$$

The maximal interval of existence (it should always contain the initial point $x = 0$) is $\boxed{(-\infty, 1).}$