MAT 126 Calculus B Spring 2007
Practice Midterm I — Solutions

Name: ____________________________  I.D.: __________  Section number: _______

Answer each question in the space provided and on the reverse side of the sheets. Show your work whenever possible. Unless otherwise indicated, answers without justification will get little or no partial credit! Cross out anything that grader should ignore and circle or box the final answer. The actual exam will contain 5 problems. This practice test contains more problems to give you more practice.

1. (a) (10 points) Estimate the area under the graph of \( f(x) = 16 - x^2 \) from \( x = 0 \) to \( x = 4 \) using four rectangles and right endpoints. Sketch the graph and rectangles. Is your estimate and underestimate or an overestimate?

Solution.

\[ R_4 = f(1) + f(2) + f(3) + f(4) = 15 + 12 + 7 = 34 \]

The function \( f(x) \) is concave downward, so that \( R_4 < A \), i.e., it is an underestimate (sketch the graph!)

(b) (10 points) Repeat part (a) using left endpoints.

Solution.

\[ L_4 = f(0) + f(1) + f(2) + f(3) = 50 \]

It is an overestimate, \( L_4 > A \) (sketch the graph!)

2. (a) (10 points) Evaluate integral by interpreting it as area

\[ \int_{-5}^{5} \sqrt{25 - x^2} \, dx \]

Solution. It is the upper half of the circle of radius 5 centered at the origin, so

\[ \int_{-5}^{5} \sqrt{25 - x^2} \, dx = \frac{1}{2} \pi 5^2 = 12.5 \pi \]

(b) (5 points) Determine a region whose area is equal to

\[ \lim_{n \to \infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan \frac{i\pi}{4n} \]

Do not evaluate the limit.
**Solution.** We identify $\Delta x = \pi/4n$. Since in general
\[ \Delta x = \frac{b - a}{n}, \]
we conclude that $b - a = \pi/4$. Comparing the general expression $f(x_i) = f(a + i\Delta x)$ with $\tan \frac{i\pi}{4n}$, we conclude that $f(x) = \tan x$ and $a = 0$. Thus we have the right endpoint sum for the function $f(x) = \tan x$ on the interval $[0, \frac{\pi}{4}]$, and
\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan \frac{i\pi}{4n} = \int_{0}^{\pi/4} \tan x\,dx.
\]
Thus the region is the region under the graph of $y = \tan x$ from $x = 0$ to $x = \pi/4$.

**3.** Given two functions $f(x)$ and $g(x)$ which satisfy
\[
\int_{0}^{3} f(x)\,dx = 5, \quad \int_{0}^{5} f(x)\,dx = 7,
\]
\[
\int_{3}^{5} g(x)\,dx = 1, \quad \int_{0}^{5} g(x)\,dx = 9,
\]
find
**(a) (5 points)**
\[
\int_{3}^{5} (3f(x) - g(x))\,dx
\]
**Solution.** We have
\[
\int_{3}^{5} f(x)\,dx = \int_{0}^{5} f(x)\,dx - \int_{0}^{3} f(x)\,dx = 7 - 5 = 2
\]
so that
\[
\int_{3}^{5} (3f(x) - g(x))\,dx = 3 \int_{3}^{5} f(x)\,dx - \int_{3}^{5} g(x)\,dx = 6 - 1 = 5
\]
**(b) (5 points)**
\[
\int_{0}^{3} (f(x) + 2g(x))\,dx
\]
**Solution.** Similarly,
\[
\int_{0}^{3} g(x)\,dx = 9 - 1 = 8
\]
4. (5 points) Express the limit as a definite integral on the given interval [0, 4]:

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{e^{x_i}}{1 + x_i} \Delta x
\]

Solution.

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{e^{x_i}}{1 + x_i} \Delta x = \int_{0}^{4} \frac{e^x}{1 + x} dx
\]

5. Evaluate the following indefinite integrals

(a) (5 points)

\[
\int (3 \cos x - 4 \sin x) dx
\]

Solution.

\[
\int (3 \cos x - 4 \sin x) dx = 3 \sin x + 4 \cos x + C
\]

(b) (10 points)

\[
\int \frac{\cos x}{1 - \cos^2 x} dx
\]

Solution. Using the fundamental trigonometric identity,

\[
\int \frac{\cos x}{1 - \cos^2 x} dx = \int \frac{\cos x}{\sin^2 x} dx = -\frac{1}{\sin x} + C
\]

6. Evaluate the following definite integrals

(a) (5 points)

\[
\int_{1}^{2} x^{-2} dx
\]

Solution.

\[
\int_{1}^{2} x^{-2} dx = \frac{x^{-1}}{-1} \bigg|_{1}^{2} = -\frac{1}{2} + 1 = \frac{1}{2}
\]

(b)

\[
\int_{1}^{8} \frac{x - 1}{\sqrt[3]{x^2}} dx
\]
Solution.

\[
\int_1^8 \frac{x - 1}{\sqrt{x^2}} \, dx = \int_1^8 (x^{1/2} - x^{-2/3}) \, dx \\
= \left[ \left( \frac{3}{4} x^{4/3} - 3x^{1/3} \right) \right]_1^8 = \left( \frac{3}{4} \times 8^{4/3} - 3 \times 8^{1/3} \right) - \left( \frac{3}{4} - 3 \right) = 8.25 
\]

where we have used that \( 8 = 2^3 \).

(c) (5 points)

\[
\int_1^{27} \frac{1}{9t} \, dt 
\]

Solution.

\[
\int_1^{27} \frac{1}{9t} \, dt = \frac{1}{9} \ln t \bigg|_1^{27} = \frac{1}{9} \ln 27 = \frac{1}{3} \ln 3, 
\]

where we have used that \( \ln 1 = 0 \) and \( 27 = 3^3 \).

(d) (5 points)

\[
\int_{\ln 3}^{\ln 6} 5e^x \, dx 
\]

Solution.

\[
\int_{\ln 3}^{\ln 6} 5e^x \, dx = 5e^x \bigg|_{\ln 3}^{\ln 6} = 5(e^{\ln 6} - e^{\ln 3}) = 5(6 - 3) = 15 
\]

(e) (10 points)

\[
\int_{\pi/3}^{\pi/2} \csc x \cot x \, dx 
\]

Solution.

\[
\int_{\pi/3}^{\pi/2} \csc x \cot x \, dx = \int_{\pi/3}^{\pi/2} \frac{\cos x}{\sin^2 x} \, dx \\
= \left( -\frac{1}{\sin x} \right) \bigg|_{\pi/3}^{\pi/2} = -1 + \frac{2}{\sqrt{3}}. 
\]