MAT 126 Calculus B Spring 2006
Practice Final Exam — Solutions

Answer each question in the space provided and on the reverse side of the sheets. Show your work whenever possible. Unless otherwise indicated, answers without justification will get little or no partial credit! Cross out anything that grader should ignore and circle or box the final answer. You do not need to simplify numerical answers or write their approximate values. This practice exam contains more problems than the actual test to give you more practice.

1. Evaluate the following definite integrals:
   (a) \[\int_1^9 \ln \sqrt{x} \, dx\]
   \[\int_1^9 \ln \sqrt{x} \, dx = \frac{1}{2} \int_1^9 \ln x \, dx = \frac{1}{2} (x \ln x - x)|_1^9\]
   \[= \frac{1}{2} (9 \ln 9 - 9 - (\ln 1 - 1)) = 9 \ln 3 - 4,\]
   where we have used that the antiderivative of \(\ln x\) is \(x \ln x - x\) (see Section 5.6, Example 2).
   (b) \[\int_0^2 \frac{x}{1 + 2x^2} \, dx\]
   Using the substitution \(u = 1 + 2x^2\) we get \(du = 4xdx\), so that \(xdx = \frac{1}{4} du\), and the limits of integration \(x = 0\) and \(x = 2\) correspond to \(u = 1\) and \(u = 9\). We get
   \[\int_0^2 \frac{x}{1 + 2x^2} \, dx = \frac{1}{4} \int_1^9 \frac{du}{u}\]
   \[= \frac{1}{4} \ln u|_1^9 = \frac{1}{2} \ln 3.\]
   (c) \[\int_1^e \frac{(\ln x)^3}{x} \, dx\]
   The substitution \(u = \ln x\) gives \(du = \frac{dx}{x}\) and limits of integration \(x = 1\) and \(x = e\) correspond to \(u = 0\) and \(u = 1\). We have
   \[\int_1^e \frac{(\ln x)^3}{x} \, dx = \int_0^1 u^3 du = \frac{1}{4} u^4|_0^1 = \frac{1}{4}.\]
(d) \[ \int_{-1}^{1} x^2 \sin(x^5) \, dx \]

The function \( f(x) = x^2 \sin(x^5) \) is odd, \( f(-x) = -f(x) \), so using the property of symmetric functions (see Section 5.5), we get

\[ \int_{-1}^{1} x^2 \sin(x^5) \, dx = 0. \]

(e) \[ \int_{0}^{1/2} \frac{\sin^{-1} x}{\sqrt{1 - x^2}} \, dx \]

Using the substitution \( u = \sin^{-1} x \), we get \( du = \frac{dx}{\sqrt{1 - x^2}} \), and the limits of integration \( x = 0 \) and \( x = 1/2 \) correspond to \( u = 0 \) and \( u = \pi/6 \). We get

\[ \int_{0}^{1/2} \frac{\sin^{-1} x}{\sqrt{1 - x^2}} \, dx = \int_{0}^{\pi/6} u \, du = \frac{1}{2} \left[ u^2 \right]_{0}^{\pi/6} = \frac{\pi^2}{72}. \]

(f) \[ \int_{1}^{4} \sqrt{t} \ln t \, dt \]

Here we use integration by parts with \( u = \ln t \) and \( dv = \sqrt{t} \). We have \( du = \frac{dt}{t} \) and \( v = \frac{2}{3} t^{3/2} \), so that

\[ \int_{1}^{4} \sqrt{t} \ln t \, dt = \int_{1}^{4} u \, dv = uv |_{1}^{4} - \int_{1}^{4} v \, du \\
= \frac{2}{3} (16 \ln 2) - \frac{2}{3} \int_{1}^{4} \sqrt{t} \, dt \\
= \frac{32}{3} \ln 2 - \frac{4}{9} t^{3/2} |_{1}^{4} = \frac{32}{3} \ln 2 - \frac{28}{9}. \]

(g) \[ \int_{0}^{13} \frac{dx}{\sqrt{3(1 + 2x)^2}} \]

We use the substitution \( u = 1 + 2x \), so that \( du = 2 \, dx \) and the limits of integration \( x = 0 \) and \( x = 13 \) correspond to \( u = 1 \) and \( u = 27 \). We get

\[ \int_{0}^{13} \frac{dx}{\sqrt{(1 + 2x)^2}} = \frac{1}{2} \int_{1}^{27} u^{-\frac{3}{2}} \, du = \frac{1}{2} \left[ 3u^{\frac{1}{2}} \right]_{1}^{27} = 3. \]
(h) \[ \int_{0}^{\frac{\pi}{2}} \sin^3 x \, dx \]

This is a trigonometric integral. Writing
\[ \sin^3 x = \sin^2 x \sin x = (1 - \cos^2 x) \sin x, \]
we recognize the substitution \( u = \cos x \). We have \( du = - \sin x \, dx \), and the limits of integration \( x = 0 \) and \( x = \frac{\pi}{2} \) correspond to \( u = 1 \) and \( u = 0 \). We get
\[ \int_{0}^{\frac{\pi}{2}} \sin^3 x \, dx = - \int_{1}^{0} (1 - u^2) \, du = \int_{0}^{1} (1 - u^2) \, du = \left( u - \frac{u^3}{3} \right) \bigg|_{0}^{1} = \frac{2}{3}. \]

2. Evaluate the following indefinite integrals:

(a) \[ \int x^2 e^x \, dx \]

This is Example 3 in Section 5.6.

(b) \[ \int \frac{2x^3 + 1}{x^2 + 1} \, dx \]

Doing long division, or simplifying as follows:
\[ \frac{2x^3 + 1}{x^2 + 1} = \frac{(2x^3 + 2x) + 1 - 2x}{x^2 + 1} = 2x + \frac{1 - 2x}{x^2 + 1}, \]
we get
\[ \int \frac{2x^3 + 1}{x^2 + 1} \, dx = \int \left( 2x + \frac{1 - 2x}{x^2 + 1} \right) \, dx \]
\[ = x^2 + \int \frac{1}{x^2 + 1} \, dx - 2 \int \frac{x}{x^2 + 1} \, dx \]
\[ = x^2 + \tan^{-1} x - \ln(x^2 + 1) + C, \]
where in the last integral we have used the substitution \( u = x^2 + 1 \).

(c) \[ \int \frac{\tan^{-1} x}{1 + x^2} \, dx \]

Using the substitution \( u = \tan^{-1} x \), we get \( du = \frac{dx}{1 + x^2} \), so that
\[ \int \frac{\tan^{-1} x}{1 + x^2} \, dx = \int u \, du = \frac{1}{2} u^2 + C = \frac{1}{2} \left( \tan^{-1} x \right)^2 + C. \]
(d) \[
\int \sin^{-1} x \, dx
\]

Using integration by parts with \( u = \sin^{-1} x \) and \( dv = dx \) we get

\[
du = \frac{dx}{\sqrt{1 - x^2}} \quad \text{and} \quad v = x,
\]

so that

\[
\int \sin^{-1} x \, dx = \int u \, dv = uv - \int v \, du
\]

\[
= x \sin^{-1} x - \int \frac{x}{\sqrt{1 - x^2}} \, dx.
\]

To compute the last integral, use the substitution \( u = 1 - x^2 \), so that \( du = -2x \, dx \) and

\[
\int \frac{x}{\sqrt{1 - x^2}} \, dx = -\frac{1}{2} \int \frac{du}{\sqrt{u}} = -\sqrt{u} + C = -\sqrt{1 - x^2} + C.
\]

Thus, finally,

\[
\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1 - x^2} + C.
\]

(e) \[
\int \frac{x - 1}{x^2 + 3x + 2} \, dx
\]

Using partial fractions,

\[
\frac{x - 1}{x^2 + 3x + 2} = \frac{x - 1}{(x + 1)(x + 2)} = \frac{A}{x + 1} + \frac{B}{x + 2},
\]

where \( A \) and \( B \) are such that

\[
x - 1 = A(x + 2) + B(x + 1)
\]

holds for all \( x \). Setting \( x = -1 \), we get \( A = -2 \), and setting \( x = -2 \), we get \( B = 3 \). Thus

\[
\int \frac{x - 1}{x^2 + 3x + 2} \, dx = \int \left( -\frac{2}{x + 1} + \frac{3}{x + 2} \right) \, dx
\]

\[
= -\log(x + 1)^2 + \log|x + 2|^3 + C.
\]

(f) \[
\int t^2 \cos(1 - t^3) \, dt
\]
Using the substitution $u = 1 - t^3$ we get $du = -3t^2 dt$ and
\[
\int t^2 \cos(1-t^3) \, dt = -\frac{1}{3} \int \cos u \, du = -\frac{1}{3} \sin u + C = -\frac{1}{3} \sin(1-t^3) + C.
\]

(g) \[
\int e^x \sqrt{1 + e^x} \, dx
\]
Using the substitution $u = 1 + e^x$ we get $du = e^x \, dx$ and
\[
\int e^x \sqrt{1 + e^x} \, dx = \int u^{\frac{1}{3}} \, du = \frac{3}{4} u^{\frac{4}{3}} + C = \frac{3}{4} (1 + e^x)^{\frac{4}{3}} + C.
\]

(h) \[
\int \cos^5 x \, dx
\]
This is a trigonometric integral. Writing
\[
\cos^5 x = \cos^4 x \cos x = (1 - \sin^2 x)^2 \cos x,
\]
we recognize the substitution $u = \sin x$. We have $du = \cos x \, dx$ and
\[
\int \cos^5 x \, dx = \int (1 - \sin^2 x)^2 \cos x \, dx = \int (1 - u^2)^2 \, du = \int (1 - 2u^2 + u^4) \, du = u - \frac{2}{3} u^3 + \frac{1}{5} u^5 + C = \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + C.
\]

3. (a) Write a formula for $\cos^2 x$ in terms of $\sin^2 x$.
\[
\cos^2 x = 1 - \sin^2 x.
\]

(b) Evaluate
\[
\int \cos^3 x \sin^2 x \, dx
\]
\[
\int \cos^3 x \sin^2 x \, dx = \int (1 - \sin^2 x) \sin^2 x \cos x \, dx,
\]
and using the substitution $u = \sin x$, we get $du = \cos x \, dx$ and
\[
\int \cos^3 x \sin^2 x \, dx = \int u^2 (1 - u^2) \, du = \int (u^2 - u^4) \, du = \frac{u^3}{3} - \frac{u^5}{5} + C.
\]
\[
= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.
\]
4. Let

\[ f(x) = \int_2^{\sqrt{x}} \frac{\sin t}{t} dt + x^2 \]

(a) Find \( f'(x) \).

Using the chain rule with \( u = \sqrt{x} \) and the Fundamental Theorem of Calculus, we get

\[
\frac{df}{dx}(x) = \frac{d}{du} \left( \int_2^{\sqrt{x}} \frac{\sin t}{t} dt \right) \frac{du}{dx} \bigg|_{u=\sqrt{x}} + 2x
\]

\[
= \frac{\sin \sqrt{x}}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} + 2x = \frac{\sin \sqrt{x}}{2x} + 2x.
\]

(b) Evaluate \( f(4) \).

\[ f(4) = \int_2^{\sqrt{4}} \frac{\sin t}{t} dt + 4^2 = \int_2^{2} \frac{\sin t}{t} dt + 16 = 16. \]

5. Find a function \( f \) and a number \( a \) such that for \( x \),

\[ 1 + \int_a^x tf(t) dt = x^3 \]

Setting in the equation \( x = a \), we get \( 1 = a^3 \), so that \( a = 1 \).

Differentiating both sides of the equation with respect to \( x \) and using the Fundamental Theorem of Calculus, we get

\[ xf(x) = 3x^2, \]

so that \( f(x) = 3x \).

6. (a) Let

\[ I = \int_0^4 e^{x^2} dx \]

For any value of \( n \) list the numbers \( L_n, R_n, M_n, T_n \) and \( I \) in increasing order.

The function \( f(x) = e^{x^2} \) is increasing and concave upward on the real line (check it using the second derivative test). From the graph (sketch it!) we get

\[ L_n < M_n < I < T_n < R_n. \]

(Here we used the analog of Fig. 5 on p. 419 (sketch it!) to get the relation \( M_n < I < T_n \).)

(b) Repeat part (a) for

\[ I = \int_0^{\sqrt{2}/2} e^{-x^2} dx \]
The function \( f(x) = e^{-x^2} \) is decreasing and concave downward when \( 0 \leq x \leq \sqrt{\frac{2}{2}} \) (check it using the second derivative test). From the graph and analog of Fig. 5 (sketch them!) we get

\[ R_n < T_n < I < M_n < L_n. \]

7. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

(a) \[ \int_0^\infty e^{-x} \, dx \]

The integral \( \int_0^t e^{-x} \, dx \) exists for every number \( t \geq 0 \) and

\[
\lim_{t \to \infty} \int_0^t e^{-x} \, dx = \lim_{t \to \infty} (-e^{-x})_0^t = \lim_{t \to \infty} (1 - e^{-t}) = 1.
\]

The improper integral of Type 1 is convergent and \( \int_0^\infty e^{-x} \, dx = 1. \)

(b) \[ \int_0^1 \frac{1}{\sqrt{x}} \, dx \]

The integral \( \int_t^1 \frac{1}{\sqrt{x}} \, dx \) exists for every number \( t > 0 \) and

\[
\lim_{t \to 0} \int_t^1 \frac{1}{\sqrt{x}} \, dx = \lim_{t \to 0} (2\sqrt{x})_t^1 = 2 \lim_{t \to 0} (1 - \sqrt{t}) = 2.
\]

The improper integral of Type 2 is convergent and \( \int_0^1 \frac{1}{\sqrt{x}} \, dx = 2. \)

(c) \[ \int_0^3 \frac{1}{x\sqrt{x}} \, dx \]
The integral \( \int_{t}^{3} \frac{1}{x\sqrt{x}} \, dx \) exists for every number \( t > 0 \) and

\[
\lim_{t \to 0} \int_{t}^{3} \frac{1}{x\sqrt{x}} \, dx = \lim_{t \to 0} (-2x^{-1/2})|_{t}^{3} = -\frac{2}{\sqrt{3}} + 2 \lim_{t \to 0} \frac{1}{\sqrt{t}}.
\]

The last limit is \( \infty \) — does not exist as a finite number, so that improper integral of Type 2 is divergent.

(d) \( \int_{\infty}^{\infty} xe^{-x^2} \, dx \)

Since

\[
\int_{-\infty}^{\infty} xe^{-x^2} \, dx = \int_{-\infty}^{0} xe^{-x^2} \, dx + \int_{0}^{\infty} xe^{-x^2} \, dx,
\]

we must evaluate both integrals separately. Using the substitution \( u = x^2 \) (one could also use \( u = -x^2 \)), we get \( du = 2xdx \) and

\[
\int_{0}^{\infty} xe^{-x^2} \, dx = \lim_{t \to \infty} \int_{0}^{t} xe^{-x^2} \, dx = \frac{1}{2} \lim_{t \to \infty} \int_{0}^{t^2} e^{-u} \, du = \frac{1}{2} \lim_{t \to \infty} (-e^{-u})|_{0}^{t} = \frac{1}{2} \lim_{t \to \infty} (1 - e^{-t^2}) = \frac{1}{2}.
\]

Using this result and the fact that \( xe^{-x^2} \) is an odd function we get

\[
\int_{-\infty}^{0} xe^{-x^2} \, dx = \lim_{t \to -\infty} \int_{t}^{0} xe^{-x^2} \, dx = -\lim_{-t \to \infty} \int_{0}^{-t} xe^{-x^2} \, dx = -\frac{1}{2}.
\]

Thus the improper integral of Type 1 is convergent and

\[
\int_{-\infty}^{\infty} xe^{-x^2} \, dx = \frac{1}{2} + \left(-\frac{1}{2}\right) = 0.
\]

(e) \( \int_{0}^{1} \frac{1}{4y - 1} \, dy \)
The integrand is \( f(y) = \frac{1}{4y-1} \), and it is discontinuous (blows up) at \( y = \frac{1}{4} \). Thus

\[
\int_{0}^{1} \frac{1}{4y-1} \, dy = \int_{0}^{\frac{1}{4}} \frac{1}{4y-1} \, dy + \int_{\frac{1}{4}}^{1} \frac{1}{4y-1} \, dy,
\]

and we need to investigate both improper integrals of Type 2. For the first integral we have, using the substitution \( u = 4y - 1, \, du = 4 \, dy \),

\[
\int_{0}^{\frac{1}{4}} \frac{1}{4y-1} \, dy = \lim_{t \to \frac{1}{4}} \int_{0}^{t} \frac{1}{4y-1} \, dy = \frac{1}{4} \lim_{t \to \frac{1}{4}} \int_{-1}^{4t-1} \frac{1}{u} \, du = \frac{1}{4} \lim_{t \to \frac{1}{4}} \ln |u|_{-1}^{4t-1} = \frac{1}{4} \lim_{t \to \frac{1}{4}} (\ln |4t-1| - \ln |-1|) = \frac{1}{4} \lim_{t \to \frac{1}{4}} \ln |4t-1| = -\infty,
\]

since \( \ln 0 = -\infty \). Thus the first improper integral is divergent, so that the integral in question is also divergent.

8. Find the area of the region bounded by the curves:
   (a) \( y = x^2 \) and \( y = x^4 \).
   The curves intersect at the points \( x = -1, 0, 1 \) and the top and bottom boundaries of the enclosed region are \( y = x^2 \) and \( y = x^4 \) (sketch the graph!). We have
   \[
   A = \int_{-1}^{1} (x^2 - x^4) \, dx = 2 \int_{0}^{1} (x^2 - x^4) \, dx = 2 \left( \frac{x^3}{3} - \frac{x^5}{5} \right) \bigg|_{0}^{1} = \frac{4}{15}.
   \]

   (b) \( x + y^2 = 2 \) and \( x + y = 0 \).
   The curves intersect at the points with coordinates \((-2, 2)\) and \((1, -1)\), and the top and bottom boundaries of the enclosed region are \( x = 2 - y^2 \) and \( x = -y \), where we are using \( y \) as an independent variable (sketch the graph!). We have
   \[
   A = \int_{-1}^{2} (2 - y^2 - (-y)) \, dy = \int_{-1}^{2} (2 - y^2 + y) \, dy = \left( 2y - \frac{y^3}{3} + \frac{y^2}{2} \right) \bigg|_{-1}^{2} = \frac{4}{2}.
   \]
9. (a) Find the volume of the solid of revolution obtained by rotating the region bounded by the curves \( y = x^2 \) and \( y^2 = x \) about the \( x \)-axis.

The curves \( y = x^2 \) and \( y = \sqrt{x} \) (we solved the second equation for \( y \), which is assumed to be positive) intersect at \( x = 0 \) and \( x = 1 \). The region has the curve \( y = \sqrt{x} \) as the top boundary and the curve \( y = x^2 \) as the bottom boundary (sketch the graph!). A cross-section is a washer with the inner radius \( x^2 \) and the outer radius \( \sqrt{x} \). The cross-sectional area is \( A(x) = \pi(x - x^4) \), and the volume of the solid of revolution is

\[
V = \int_0^1 A(x) \, dx = \pi \int_0^1 (x - x^4) \, dx = \pi \left( \frac{x^2}{2} - \frac{x^5}{5} \right) \bigg|_0^1 = 3\pi/10.
\]

(b) Find the volume of the solid of revolution obtained by rotating the region bounded by \( y = \sec x \), \( y = 1 \), \( x = -1 \) and \( x = 1 \) about the \( x \)-axis.

The region has the horizontal line \( y = \sec x \) as the top boundary, the curve \( y = 1 \) as the bottom boundary, and the lines \( x = -1 \) and \( x = 1 \) as the vertical boundaries (sketch the graph!). A cross-section is a washer with the inner radius 1 and the outer radius \( \sec x \). The cross-sectional area is \( A(x) = \pi(\sec^2 x - 1) \), and the volume of the solid of revolution is

\[
V = \int_{-1}^1 A(x) \, dx = \pi \int_{-1}^1 (\sec^2 x - 1) \, dx = 2\pi \int_0^1 (\sec^2 x - 1) \, dx = 2\pi (\tan x - x) \bigg|_0^1 = 2\pi (\tan 1 - 1).
\]

10. Find the length of the following curves:

(a) \( y = x^{3/2}, \ 0 \leq x \leq 2 \).

\[
L = \int_0^2 \sqrt{1 + (y')^2} \, dx = \int_0^2 \sqrt{1 + \frac{3}{2}x} \, dx.
\]

Using the substitution \( u = 1 + \frac{3}{2}x \), we get

\[
L = \frac{4}{9} \int_1^{11/2} \sqrt{u} \, du = \frac{4}{9} \cdot \frac{2}{3} u^{3/2} \bigg|_1^{11/2} = \frac{8}{27} \left( \left( \frac{11}{2} \right)^{3/2} - 1 \right).
\]
(b)  
\[ y = \frac{x^2}{4} - \frac{\ln x}{2}, \quad 1 \leq x \leq 2 \]

\[ L = L = \int_1^2 \sqrt{1 + (y')^2} \, dx = \int_1^2 \sqrt{1 + \frac{1}{4} \left( x - \frac{1}{x} \right)^2} \, dx. \]

We have, by simple algebra,

\[ 1 + \frac{1}{4} \left( x - \frac{1}{x} \right)^2 = 1 + \frac{1}{4} (x^2 - 2 + x^{-2}) = \frac{1}{4} (x^2 + x^{-2}) + \frac{1}{2} \]

\[ = \frac{1}{4} (x^2 + 2 + x^{-2}) = \frac{1}{4} (x + x^{-1})^2. \]

Thus

\[ L = \int_1^2 \sqrt{\frac{1}{4} (x + x^{-1})^2} \, dx = \frac{1}{2} \int_1^2 (x + x^{-1}) \, dx = \frac{1}{2} \left( \frac{x^2}{2} + \ln x \right) \bigg|_1^2 = \frac{3}{4} + \frac{\ln 2}{2}. \]

11. Find the average value \( f_{\text{ave}} \) of \( f \) on the given interval.

(a) \( f(x) = x \sin(x^2) \) on \([0, \sqrt{\pi}]\).

We get, using the substitution \( u = x^2, \, du = 2x \, dx \),

\[ f_{\text{ave}} = \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{\pi}} x \sin(x^2) \, dx = \frac{1}{2\sqrt{\pi}} \int_0^{\pi} \sin u \, du \]

\[ = \frac{1}{2\sqrt{\pi}} (-\cos u) \bigg|_0^\pi = \frac{1}{2\sqrt{\pi}} (-\cos \pi + \cos 0) = \frac{1}{\sqrt{\pi}}. \]

(b) \( f(x) = 4 - x^2 \) on \([0, 3]\).

\[ f_{\text{ave}} = \frac{1}{3} \int_0^3 (4 - x^2) \, dx = \frac{1}{3} \left( 4x - \frac{x^3}{3} \right) \bigg|_0^3 \]

\[ = \frac{1}{3} (12 - 9) = 1. \]

(c) For \( f \) as in part (b) find the number \( c \) in \([0, 3]\) such that

\[ f(c) = f_{\text{ave}}. \]

Solving \( 4 - c^2 = 1 \) we get \( c = \sqrt{3} \) as the only solution which belongs to the interval \([0, 3]\).