1. (a) (10 points) Estimate the area under the graph of \( f(x) = 16 - x^2 \) from \( x = 0 \) to \( x = 4 \) using four rectangles and right endpoints. Sketch the graph and rectangles. Is your estimate an underestimate or an overestimate?

Solution.

\[
R_4 = f(1) + f(2) + f(3) + f(4) = 15 + 12 + 7 = 34
\]

The function \( f(x) \) is concave downward, so that \( R_4 < A \), i.e., it is an underestimate (sketch the graph!)

(b) (10 points) Repeat part (a) using left endpoints.

Solution.

\[
L_4 = f(0) + f(1) + f(2) + f(3) = 50
\]

It is an overestimate, \( L_4 > A \) (sketch the graph!)

2. (a) (10 points) Evaluate integral by interpreting it as area

\[
\int_{-5}^{5} \sqrt{25 - x^2} \, dx
\]

Solution. It is the upper half of the circle of radius 5 centered at the origin, so

\[
\int_{-5}^{5} \sqrt{25 - x^2} \, dx = \frac{1}{2} \pi 5^2 = 12.5 \pi
\]

(b) (5 points) Determine a region whose area is equal to

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan \frac{i\pi}{4n}
\]

Do not evaluate the limit.
Solution. It is the right endpoint sum for the function $f(x) = \tan x$ on the interval $[0, \frac{\pi}{4}]$. Thus

$$\lim_{n \to \infty} \sum_{i=1}^{n} \pi \tan \left( \pi \frac{i}{4n} \right) = \int_{0}^{\pi/4} \tan x \, dx$$

3. Given two functions $f(x)$ and $g(x)$ which satisfy

$$\int_{0}^{3} f(x) \, dx = 5, \quad \int_{0}^{5} f(x) \, dx = 7,$$
$$\int_{3}^{5} g(x) \, dx = 1, \quad \int_{0}^{5} g(x) \, dx = 9,$$

find

(a) (5 points)

$$\int_{3}^{5} (3f(x) - g(x)) \, dx$$

Solution. We have

$$\int_{3}^{5} f(x) \, dx = \int_{0}^{5} f(x) \, dx - \int_{0}^{3} f(x) \, dx = 7 - 5 = 2$$

so that

$$\int_{3}^{5} (3f(x) - g(x)) \, dx = 3 \int_{3}^{5} f(x) \, dx - \int_{3}^{5} g(x) \, dx = 6 - 1 = 5$$

(b) (5 points)

$$\int_{0}^{3} (f(x) + 2g(x)) \, dx$$

Solution. Similarly,

$$\int_{0}^{3} g(x) \, dx = 9 - 1 = 8$$

and

$$\int_{0}^{3} (f(x) + 2g(x)) \, dx = 5 + 2 \times 8 = 21$$

4. (5 points) Express the limit as a definite integral on the given interval $[0, 4]$:

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{e^{x_i}}{1 + x_i} \Delta x$$
Solution.
\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{e^{x_i}}{1 + x_i} \Delta x = \int_0^4 \frac{e^x}{1 + x} dx
\]

5. Evaluate the following indefinite integrals
(a) (5 points)
\[
\int (3 \cos x - 4 \sin x) dx
\]
Solution.
\[
\int (3 \cos x - 4 \sin x) dx = 3 \sin x + 4 \cos x + C
\]
(b) (10 points)
\[
\int \frac{\cos x}{1 - \cos^2 x} dx
\]
Solution. Using the fundamental trigonometric identity,
\[
\int \frac{\cos x}{1 - \cos^2 x} dx = \int \frac{\cos x}{\sin^2 x} dx = -\frac{1}{\sin x} + C
\]

6. Evaluate the following definite integrals
(a) (5 points)
\[
\int_1^2 x^{-2} dx
\]
Solution.
\[
\int_1^2 x^{-2} dx = \left. \frac{x^{-1}}{-1} \right|_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}
\]
(b)
\[
\int_1^8 \frac{x - 1}{\sqrt[3]{x^2}} dx
\]
Solution.
\[
\int_1^8 \frac{x - 1}{\sqrt[3]{x^2}} dx = \int_1^8 (x^{1/3} - x^{-2/3}) dx
\]
\[
= \left( \frac{3}{4} x^{4/3} - 3x^{1/3} \right) \bigg|_1^8 = \left( \frac{3}{4} \times 8^{4/3} - 3 \times 8^{1/3} \right) - \left( \frac{3}{4} - 3 \right) = 8.25
\]
where we have used that 8 = 2³.
(c) (5 points)
\[ \int_{1}^{27} \frac{1}{9t} \, dt \]

Solution.
\[ \int_{1}^{27} \frac{1}{9t} \, dt = \frac{1}{9} \ln t \bigg|_{1}^{27} = \frac{1}{9} \ln 27 = \frac{1}{3} \ln 3 \]

, where we have used that \( \ln 1 = 0 \) and \( 27 = 3^3 \).

(d) (5 points)
\[ \int_{\ln 3}^{\ln 6} 5e^x \, dx \]

Solution.
\[ \int_{\ln 3}^{\ln 6} 5e^x \, dx = 5e^x \bigg|_{\ln 3}^{\ln 6} = 5(e^{\ln 6} - e^{\ln 3}) = 5(6 - 3) = 15 \]

(e) (10 points)
\[ \int_{\pi/3}^{\pi/2} \csc x \cot x \, dx \]

Solution.
\[ \int_{\pi/3}^{\pi/2} \csc x \cot x \, dx = \int_{\pi/3}^{\pi/2} \frac{\cos x}{\sin^2 x} \, dx \]
\[ = \left( - \frac{1}{\sin x} \right) \bigg|_{\pi/3}^{\pi/2} = -1 + \frac{2}{\sqrt{3}} \]