MAT 126 Calculus B Fall 2005
Practice Midterm II Solutions

Answer each question in the space provided and on the reverse side of the sheets. Show your work whenever possible. Unless otherwise indicated, answers without justification will get little or no partial credit! Cross out anything that grader should ignore and circle or box the final answer. The actual exam will contain 5 problems. This practice test contains more problems to give you more practice.

1. Evaluate the following definite integrals

(a) \[ \int_{0}^{13} \frac{2}{(2x+1)^{\frac{3}{2}}} \, dx \]

Solution. Substitution \( u = 2x + 1 \) gives \( du = 2dx \) and \( x = 0 \) corresponds to \( u = 1 \) and \( x = 13 \) — to \( u = 27 \). Thus by the substitution rule,
\[
\int_{0}^{13} \frac{2}{(2x+1)^{\frac{3}{2}}} \, dx = \left[ \frac{1}{u^{\frac{1}{2}}} \right]_{1}^{27} = 3(27)^{\frac{1}{3}} - 3 = 3 \cdot 3 - 3 = 6.
\]

(b) \[ \int_{0}^{\frac{\pi}{2}} e^{\sin x} \cos x \, dx \]

Solution. Substitution \( u = \sin x \) gives \( du = \cos x \, dx \) and \( x = 0 \) corresponds to \( u = 0 \) and \( x = \frac{\pi}{2} \) — to \( u = 1 \). Thus by the substitution rule,
\[
\int_{0}^{\frac{\pi}{2}} e^{\sin x} \cos x \, dx = \int_{0}^{1} e^{u} \, du = e^{1} - e^{0} = e - 1.
\]

(c) \[ \int_{0}^{1} x^{4}(1 + x^{5})^{20} \, dx \]

Solution. Substitution \( u = 1 + x^{5} \) gives \( du = 5x^{4} \, dx \) and \( x = 0 \) corresponds to \( u = 1 \) and \( x = 1 \) — to \( u = 2 \). Thus by the substitution rule,
\[
\int_{0}^{1} x^{4}(1 + x^{5})^{20} \, dx = \frac{1}{5} \int_{1}^{2} u^{20} \, du = \left[ \frac{u^{21}}{5 \cdot 21} \right]_{1}^{2} = \frac{2^{21} - 1}{105}.
\]

(d) \[ \int_{0}^{1} \tan^{-1} x \, dx \]
Solution. We use integration by parts with $u = \tan^{-1} x$ and $dv = dx$. We have $du = \frac{dx}{1 + x^2}$ and $v = x$, so that using $\tan^{-1}(1) = \frac{\pi}{4}$, we get

$$
\int_0^1 \tan^{-1} x \, dx = \int_0^1 udv = uv\Big|_0^1 - \int_0^1 vdu = \frac{\pi}{4} - \int_0^1 \frac{x}{1 + x^2} \, dx.
$$

To evaluate the remaining integral, we use the substitution $u = 1 + x^2$, so that $du = 2xdx$ and $x = 0$ corresponds to $u = 1$ and $x = 1$ — to $u = 2$. Thus

$$
\int_0^1 \frac{x}{1 + x^2} \, dx = \frac{1}{2} \int_1^2 \frac{du}{u} = \frac{\ln u}{2} \Big|_1^2 = \frac{\ln 2}{2}.
$$

Therefore, we get

$$
\int_0^1 \tan^{-1} x \, dx = \frac{\pi}{4} - \frac{\ln 2}{2}.
$$

\(e\)

$$
\int_0^{\pi/2} \cos^5 t \, dt.
$$

Solution. This is a trigonometric integral, so we use the main trigonometric identity and represent

$$
\cos^5 t = \cos^4 t \cdot \cos t = (1 - \sin^2 t)^2 \cos t.
$$

Now the substitution $u = \sin t$ gives $du = \cos t \, dt$, and $t = 0$ corresponds to $u = 0$ and $t = \frac{\pi}{2}$ — to $u = 1$. We have

$$
\int_0^{\pi/2} \cos^5 t \, dt = \int_0^{\pi/2} (1 - \sin^2 t)^2 \cos t \, dt
$$

$$
= \int_0^1 (1 - u^2)^2 \, du = \int_0^1 (1 - 2u^2 + u^4) \, du
$$

$$
= \left( u - \frac{2u^3}{3} + \frac{u^5}{5} \right)\bigg|_0^1 = \frac{8}{15}.
$$

\(f\)

$$
\int_0^{\pi/2} \frac{\sin^{-1} x}{\sqrt{1 - x^2}} \, dx
$$
Solution. Substitution \( u = \sin^{-1} x \) gives \( du = \frac{dx}{\sqrt{1-x^2}} \), and \( x = 0 \) corresponds to \( u = 0 \) and \( x = \frac{1}{2} \) to \( u = \frac{\pi}{6} \), so that

\[
\int_{0}^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}} \, dx = \int_{0}^{\pi/6} u \, du = \frac{\pi^2}{72}.
\]

(g)

\[
\int_{-\pi}^{\pi} \frac{x^2 \sin^3 x}{1 + x^6} \, dx
\]

Solution. This is the integral over symmetric interval \([-\pi, \pi]\), and the function

\[
f(x) = \frac{x^2 \sin^3 x}{1 + x^6}
\]

is odd, \( f(-x) = -f(x) \), so that

\[
\int_{-\pi}^{\pi} \frac{x^2 \sin^3 x}{1 + x^6} \, dx = 0
\]

2. Evaluate the following indefinite integrals

(a)

\[
\int x^3 e^{x^4} \, dx
\]

Solution. Setting \( u = x^4 \), we get \( du = 4x^3 \, dx \), so by the substitution rule,

\[
\int x^3 e^{x^4} \, dx = \frac{1}{4} \int e^u \, du = \frac{1}{4} e^u + C = \frac{1}{4} e^{x^4} + C.
\]

(b)

\[
\int te^t \, dt
\]

Solution. We use integration by parts with \( u = t \) and \( dv = e^t \, dt \). We have \( du = dt \) and \( v = e^t \), so that

\[
\int te^t \, dt = \int u \, dv = uv - \int v \, du = te^t - e^t - C.
\]

(c)

\[
\int x^2 \cos x \, dx
\]

Solution. We use integration by parts with \( u = x^2 \) and \( dv = \cos x \, dx \). We have \( du = 2x \, dx \) and \( v = \sin x \), so that

\[
\int x^2 \cos x \, dx = \int u \, dv = uv - \int v \, du = x^2 \sin x - 2 \int x \sin x \, dx.
\]
For the remaining integral we again use integration by parts with \( u = 2x \) and \( dv = \sin x \, dx \), so that \( du = 2 \, dx \) and \( v = -\cos x \). We have

\[
2 \int x \sin x \, dx = \int u \, dv = uv - \int v \, du = -2x \cos x + 2 \int \cos x \, dx
\]

so that

\[
\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C.
\]

(Double-check the answer by differentiating!)

(d) \[
\int \cos(\sqrt{x}) \, dx
\]

Solution. First, we use the substitution rule with \( t = \sqrt{x} \), so that \( dt = \frac{1}{2\sqrt{x}} \, dx \), or \( dx = 2\sqrt{x} \, dt = 2t \, dt \). We get

\[
\int \cos(\sqrt{x}) \, dx = 2 \int t \cos(t) \, dt.
\]

To evaluate this integral, we use integration by parts with \( u = 2t \) and \( dv = \cos(t) \, dt \). We have \( du = 2 \, dt \) and \( v = \sin t \), so that

\[
2 \int t \cos(t) \, dt = \int u \, dv = uv - \int v \, du = 2t \sin t - 2 \int \sin t \, dt = 2t \sin t + 2 \cos t + C.
\]

Finally, remembering that \( t = \sqrt{x} \), we get

\[
\int \cos(\sqrt{x}) \, dx = 2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} + C.
\]

3. Evaluate the following indefinite integrals

(a) \[
\int \frac{1}{x^2} \ln x \, dx
\]

Solution. Here we use integration by parts with \( u = \ln x \) and \( dv = \frac{1}{x^2} \, dx \), so that

\[
du = \frac{1}{x} \, dx \quad \text{and} \quad v = -\frac{1}{x}.
\]
(Note that substitution rule with \( u = \ln x \) does not simplify the integral since in the denominator we have \( x^2 \); if it was \( x \), then the substitution rule would work.) Thus we have

\[
\int \frac{1}{x^2} \ln x \, dx = \int u \, dv = uv - \int v \, du = -\frac{\ln x}{x} + \int \frac{1}{x^2} \, dx = -\frac{\ln x}{x} - \frac{1}{x} + C.
\]

(b) \[
\int \frac{1}{x} (\ln x)^2 \, dx
\]

Solution. Here we use the substitution rule with \( u = \ln x \) and \( du = \frac{1}{x} \, dx \) (since we have \( x \) in the denominator). Therefore,

\[
\int \frac{1}{x} (\ln x)^2 \, dx = \int u^2 \, du = \frac{1}{3} u^3 + C = \frac{1}{3} (\ln x)^3 + C.
\]

(c) \[
\int x^3 \ln x \, dx
\]

Solution. Here we use integration by parts with \( u = \ln x \) and \( dv = x^3 \, dx \), so that

\[
du = \frac{1}{x} \, dx \quad \text{and} \quad v = \frac{2}{5} x^{5/2}.
\]

Thus we have

\[
\int x^3 \ln x \, dx = \int u \, dv = uv - \int v \, du = \frac{2}{5} x^{5/2} \ln x - \frac{2}{5} \int x^{3/2} \, dx = \frac{2}{5} x^{5/2} \ln x - \frac{4}{25} x^{5/2} + C.
\]

4. Evaluate the following indefinite integrals

(a) \[
\int \frac{2x^2}{x^2 + 1} \, dx
\]

Solution. We have

\[
\frac{2x^2}{x^2 + 1} = 2 - \frac{2}{x^2 + 1}
\]

(either by doing the long division, or by writing \( 2x^2 = 2x^2 + 2 - 2 = 2(x^2 + 1) - 2 \), and dividing both terms by
Thus
\[ \int \frac{2x^2}{x^2 + 1} \, dx = \int \left( 2 - \frac{2}{x^2 + 1} \right) \, dx = 2x - 2 \tan^{-1} x + C. \]

(b)
\[ \int \frac{2}{x^2 - 1} \, dx \]

Solution. Using partial fractions, we write
\[ \frac{2}{x^2 - 1} = \frac{1}{(x - 1)(x + 1)} = \frac{A}{x - 1} + \frac{B}{x + 1}. \]
This decomposition is valid if and only if for all \( x \)
\[ 2 = A(x + 1) + B(x - 1). \]
Setting \( x = 1 \) we get \( A = 1 \), and setting \( x = -1 \) we get \( B = -1 \). (Another way to solve for \( A \) and \( B \) is to rewrite the above equation as
\[ 2 = (A + B)x + (A - B), \]
which is equivalent to the following system of linear algebraic equations for \( A \) and \( B \):
\[ A + B = 0, \]
\[ A - B = 2. \]
The solution of this system is, as before, \( A = 1, B = -1. \)
Therefore we have
\[ \int \frac{2}{x^2 - 1} \, dx = \int \left( \frac{1}{x - 1} - \frac{1}{x + 1} \right) \, dx \]
\[ = \ln |x - 1| - \ln |x + 1| + C = \ln \left| \frac{x - 1}{x + 1} \right| + C. \]

(c)
\[ \int \frac{2x}{x^2 + 1} \, dx \]

Solution. To evaluate this integral, we use the substitution
\( u = 1 + x^2 \), so that \( du = 2xdx \) (compare with the last integral in problem 2 (d)). We have
\[ \int \frac{2x}{x^2 + 1} \, dx = \int \frac{du}{u} = \ln |u| + C = \ln(1 + x^2) + C. \]
(d) \[ \int \frac{4x + 7}{2x^2 + 7x - 15} \, dx \]

*Solution.* We have (either by guessing or using the formula for the roots of quadratic polynomial)

\[ 2x^2 + 7x - 15 = (2x - 3)(x + 5) \]

so that we use method of partial fractions,

\[ \frac{4x + 7}{2x^2 + 7x - 15} = \frac{A}{2x - 3} + \frac{B}{x + 5} = \frac{A(x + 5) + B(2x - 3)}{(2x - 3)(x + 5)}. \]

Clearing the denominators, we get the equation

\[ 4x + 7 = A(x + 5) + B(2x - 3) \]

and setting \( x = -5 \) we get \( -13 = -13B \), so \( B = 1 \); setting \( x = \frac{3}{2} \) we get \( 13 = 13A/2 \), so that \( A = 2 \). As the result,

\[ \int \frac{4x + 7}{2x^2 + 7x - 15} \, dx = \int \left( \frac{2}{2x - 3} + \frac{1}{x + 5} \right) \, dx \]

\[ = \ln |2x - 3| + \ln |x + 5| + C, \]

where for the first term we used the substitution \( u = 2x - 3 \).

5. (a) Write a formula for \( \tan x \) in terms of \( \sin x \) and \( \cos x \).

*Solution.*

\[ \tan x = \frac{\sin x}{\cos x}. \]

(b) Evaluate \( \int \tan x \, dx \)

*Solution.* Using part (a) we have

\[ \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx, \]

which suggests the substitution \( u = \cos x \). We have \( du = -\sin x \, dx \), so that

\[ \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{du}{u} = -\ln |u| + C = -\ln |\cos x| + C. \]

6. Evaluate (a) \( \int \sqrt{16 - x^2} \, dx \)

*Solution.* This integral is simplified with the help of the substitution \( x = 4 \sin \theta \) (see Section 5.7). We have 16 –
\(x^2 = 16(1 - \sin^2 \theta) = 16 \cos^2 \theta\), so that \(\sqrt{16 - x^2} = 4 \cos \theta\).

Since \(dx = 4 \cos \theta d \theta\), we have
\[
\int \sqrt{16 - x^2} \, dx = 16 \int \cos^2 \theta \, d \theta .
\]

To evaluate the last integral, we use the half-angle formula
\[
\cos^2 \theta = \frac{1 + \cos 2 \theta}{2}.
\]

Thus
\[
\int \sqrt{16 - x^2} \, dx = 16 \int \cos^2 \theta \, d \theta = 8 \int (1 + \cos 2 \theta) \, d \theta
\]
\[= 8 \theta + 4 \sin 2 \theta + C
\]
\[= 8 \theta + 8 \sin \theta \cos \theta + C
\]
\[= 8 \sin^{-1} \frac{x}{4} + 2x \cos \left(\sin^{-1} \frac{x}{4}\right) + C
\]
\[= 8 \sin^{-1} \frac{x}{4} + \frac{x}{2} \sqrt{16 - x^2} + C.
\]

Here in the last three lines we have used the double angle formula for \(\sin 2 \theta\), the equation \(\theta = \sin^{-1} \frac{x}{4}\), and the formula \(\cos \theta = \sqrt{1 - \sin^2 \theta}\), which follows from the fundamental trigonometric identity.

(b)

\[
\int \frac{1}{x^2 \sqrt{x^2 + 4}} \, dx
\]

**Solution.** This integral is simplified with the help of the substitution \(x = 2 \tan \theta\) (see Section 5.7). We have, using the fundamental trigonometric identity,
\[
x^2 + 4 = 4 \tan^2 \theta + 4 = \frac{4}{\cos^2 \theta},
\]
so that \(\frac{1}{\sqrt{x^2 + 4}} = \frac{1}{2} \cos \theta\). Since \(dx = \frac{2}{\cos^2 \theta} \, d \theta\), we get
\[
\int \frac{1}{x^2 \sqrt{x^2 + 4}} \, dx = \int \frac{1}{2} \cos \theta \left(\frac{\cos^2 \theta}{4 \sin^2 \theta \cos^2 \theta} \right) \, d \theta
\]
\[= \frac{1}{4} \int \frac{\cos \theta}{\sin^2 \theta} \, d \theta = -\frac{1}{4 \sin \theta} + C
\]
\[= -\frac{1}{4 \sin(\tan^{-1} \frac{x}{2})} + C,
\]
where in the last line we have used that \(\theta = \tan^{-1}(\frac{x}{2})\).