Answer each question in the space provided and on the back of the sheets. Write full solutions, not just answers: unless otherwise marked, answers without justification will get little or no partial credit. Cross out anything the grader should ignore and circle or box the final answer. Do NOT round answers.

No books, notes, or calculators!

1. Compute the following limits by distinguishing between “lim \( f(x) = \infty \)”, “lim \( f(x) = -\infty \)”, and “limit does not exist even allowing for infinite values”.

(a) \( \lim_{x \to -1} x^3 + 7x^2 - 1 \)

Solution: Since any polynomial is continuous,
\[
\lim_{x \to -1} x^3 + 7x^2 - 1 = (-1)^3 + 7(-1)^2 - 1 = 5.
\]

(b) \( \lim_{x \to \infty} x \tan \frac{1}{x} \)

Solution: This is expression of the type “\( \infty \times 0 \)”. Since \( t = \frac{1}{x} \) goes to 0 as \( x \to \infty \), we have
\[
\lim_{x \to \infty} x \tan \frac{1}{x} = \lim_{t \to 0} \frac{\tan t}{t},
\]
so that L’Hospital’s rule is applicable. Namely, since \( (\tan t)' = \sec^2 t \),
\[
\lim_{t \to 0} \frac{\tan t}{t} = \lim_{t \to 0} \frac{\sec^2 t}{1} = 1.
\]

Note: this problem can be also solved by writing
\[
x \tan \frac{1}{x} = \frac{\tan \frac{1}{x}}{\frac{1}{x}}
\]
and using L’Hospital’s rule as \( x \to \infty \).

(c) \( \lim_{x \to 5} \frac{x^2 - 4x - 5}{x - 5} \)

Solution: We can’t just substitute \( x = 5 \), as it will give denominator zero. The numerator also becomes zero. However, factoring the numerator works:
\[
\lim_{x \to 5} \frac{x^2 - 4x - 5}{x - 5} = \lim_{x \to 5} \frac{(x + 1)(x - 5)}{x - 5} = \lim_{x \to 5} (x + 1) = 6.
\]

Note: this problem can also be solved by using L’Hospital’s rule.
(d) $\lim_{x \to 0} x^4 \cos \frac{\pi}{x}$

Solution: Since $-1 \leq \cos \frac{\pi}{x} \leq 1$, we see that

$$-x^4 \leq x^4 \cos \frac{\pi}{x} \leq x^4$$

Since $\lim_{x \to 0} x^4 = 0$, by Squeeze Theorem, $\lim_{x \to 0} x^4 \cos \frac{\pi}{x} = 0$.

(e) $\lim_{x \to 0} \frac{e^x - 1 - x - x^2/2}{x^3}$

Solution: Using L'Hospital rule three times, we have

$$\lim_{x \to 0} \frac{e^x - 1 - x - x^2/2}{x^3} = \lim_{x \to 0} \frac{e^x - 1}{3x^2} = \lim_{x \to 0} \frac{e^x - 1}{6x} = \lim_{x \to 0} \frac{e^x}{6} = \frac{1}{6}.$$

(f) $\lim_{x \to \infty} \frac{x^3 + 1001x + 77}{x^3 - x^2 + 99}$

Solution:

$$\lim_{x \to \infty} \frac{x^3 + 1001x + 77}{x^3 - x^2 + 99} = \lim_{x \to \infty} \frac{1 + \frac{1001}{x} + \frac{77}{x^3}}{1 - \frac{1}{x} + \frac{99}{x^3}} = 1 = 1.$$

(g) $\lim_{x \to \pi/2} \frac{\cos x}{2x - \pi}$

Solution: Direct substituiton $x = \pi/2$ gives $0/0$ which is meaningless. Thus, we can use L’Hospital’s rule, which gives

$$\lim_{x \to \pi/2} \frac{\cos x}{2x - \pi} = \lim_{x \to \pi/2} -\frac{\sin x}{2} = -\frac{1}{2}.$$

(h) $\lim_{x \to \infty} (xe^{1/x} - x)$

Solution: We have, after introducing the variable $t = \frac{1}{x}$ and using L’Hospital rule,

$$\lim_{x \to \infty} (xe^{1/x} - x) = \lim_{x \to \infty} x(e^{1/x} - 1) = \lim_{t \to 0} \frac{e^t - 1}{t}$$

$$= \lim_{t \to 0} \frac{e^t}{1} = 1.$$

Note: this problem can be also solved by writing

$$\lim_{x \to \infty} (xe^{1/x} - x) = \lim_{x \to \infty} \frac{e^{1/x} - 1}{1/x}$$

and using L’Hospital’s rule as $x \to \infty$. 

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2. Compute the derivatives of the following functions

(a) \( f(x) = x^3 - 12x^2 + x + 137\pi \)
   \[ f'(x) = 3x^2 - 24x + 1. \]

(b) \( f(x) = (2x + 1)\sin x \)
   \[ f'(x) = (2 + 1)'\sin x + (2 + 1)(\sin x)' = 2\sin x + (2 + 1)\cos x. \]

(c) \( g(s) = \sqrt{1 + e^{2s}} \)
   \[ \text{Solution: By chain rule, using } u = 1 + e^{2s}: \]
   \[ \frac{dg}{ds} = \frac{dg}{du} \frac{du}{ds} = \frac{d(\sqrt{u})}{du} \frac{d(1 + e^{2s})}{ds} = \frac{1}{2\sqrt{u}} \frac{2e^{2s}}{e^{2s}} = \frac{e^{2s}}{\sqrt{1 + e^{2s}}} \]

(d) \( h(t) = \frac{1 + e^t}{1 - e^t} \)
   \[ \text{Solution: By quotient rule,} \]
   \[ h'(t) = \frac{(1 + e^t)'(1 - e^t) - (1 + e^t)(1 - e^t)'}{(1 - e^t)^2} = \frac{e^t(1 - e^t) - (1 + e^t)(-e^t)}{(1 - e^t)^2} \]
   \[ = \frac{e^t - (e^t)^2 + e^t + (e^t)^2}{(1 - e^t)^2} = \frac{2e^t}{(1 - e^t)^2} \]

(e) \( f(x) = (2x + 2)^{100} \)
   \[ \text{Solution: By chain rule,} \]
   \[ f'(x) = 100(2x + 2)^{99}(2x + 2)' = 200(2x + 2)^{99} \]

(f) \( g(x) = x^{\sin x} \)
   \[ \text{Solution: We will use logarithmic derivative:} \]
   \[ (\ln g(x))' = (\ln x^{\sin x})' = (\sin x \ln x)' = (\sin x)' \ln x + (\sin x)(\ln x)' \]
   \[ = \cos x \ln x + \sin x \frac{1}{x} \]

   Thus, using \((\ln g)' = \frac{g'}{g}\), we get
   \[ g'(x) = g(x)(\ln g(x))' = x^{\sin x} \left( \cos x \ln x + \frac{\sin x}{x} \right). \]
3. Let \( f(x) = xe^{-x^2} \).

(a) Find asymptotes of \( f(x) \) (hint: \( f(x) = \frac{x}{e^{x^2}} \)).

**Solution:** This function is continuous everywhere, so there are no vertical asymptotes. To find horizontal asymptotes, we need to compute \( \lim_{x \to \pm \infty} f(x) \). Writing \( f(x) = \frac{x}{e^{x^2}} \), we see that as \( x \to \infty \), both numerator and denominator have limit \( \infty \). Thus, we can not use quotient rule (it would give \( \frac{\infty}{\infty} \), which is meaningless); however, we can use L’Hospital’s rule:

\[
\lim_{x \to \infty} \frac{x}{e^{x^2}} = \lim_{x \to \infty} \frac{1}{2xe^{x^2}} = 0
\]

since \( \lim_{x \to \infty} 2xe^{x^2} = \infty \). Similar computation gives

\[
\lim_{x \to -\infty} f(x) = 0
\]

Thus, the horizontal asymptote is \( y = 0 \).

(b) Compute the derivative of \( f(x) \)

**Solution:** \( f'(x) = (x)'e^{-x^2} + x(e^{-x^2})' = e^{-x^2} + x(-2xe^{-x^2}) = (1 - 2x^2)e^{-x^2} \)

(c) On which intervals is \( f(x) \) increasing? decreasing?

**Solution:** \( f(x) \) is increasing when \( f'(x) > 0 \), i.e. \( (1 - 2x^2)e^{-x^2} > 0 \). Since \( e^{-x^2} > 0 \), it is equivalent to \( 1 - 2x^2 > 0 \), i.e. \( 1 < 2x^2 \), or \( x^2 < 1/2 \). Solutions of this last inequality are \( -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}} \). So \( f(x) \) is increasing on \( (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \).

Same argument shows that \( f(x) \) decreasing on \( (-\infty, -\frac{1}{\sqrt{2}}) \) and on \( (\frac{1}{\sqrt{2}}, \infty) \)

(d) Sketch a graph of \( f(x) \) using the results of the previous parts and the fact that \( f(0) = 0 \).
4. Let \( f(x) = -2x^3 + 6x^2 - 3 \).

(a) Compute \( f', f'' \).

\[ f'(x) = -6x^2 + 12x \]
\[ f''(x) = -12x + 12 \]

(b) On which intervals is \( f(x) \) increasing/decreasing?

\[ f(x) \] is increasing when \( f'(x) > 0 \):

\[ -6x^2 + 12x > 0 \]
\[ -6x(x - 2) > 0 \]

Since the graph of \(-6x^2 + 12x\) is a parabola with the branches going down, this expression is positive between the roots, i.e. for \( 0 < x < 2 \). Thus, \( f'(x) > 0 \) on the interval \((0, 2)\), and \( f(x) \) is increasing on \((0, 2)\).

Similar argument shows that \( f'(x) < 0 \) on \((-\infty, 0)\) and on \((2, \infty)\); thus, on these intervals \( f(x) \) is decreasing.

(c) On which intervals is \( f(x) \) concave up/down?

\[ f(x) \] is concave up when \( f''(x) > 0 \), i.e. \(-12x + 12 > 0\), or \(1 - x > 0\), \(x < 1\). Therefore, \( f(x) \) is concave up on \((-\infty, 1)\) and concave down on \((1, \infty)\).

(d) Find all critical points of \( f(x) \). Which of them are local maximums? local minimums? neither? Justify your answer.

\[ f(x) \] critical points are where \( f''(x) = 0 \), i.e.

\[ -6x^2 + 12x = 0 \]
\[ x^2 - 2x = 0 \]
\[ x(x - 2) = 0 \]

So the critical points are \( x = 0, x = 2 \).

Since \( f(x) \) is decreasing for \( x < 0 \) and increasing for \( 0 < x < 2 \), by first derivative test, \( x = 0 \) is a local minimum. Similarly, since \( f(x) \) is increasing for \( 0 < x < 2 \) and decreasing for \( x > 2, x = 2 \) is a local maximum.

5. It is known that the polynomial \( f(x) = x^3 - x - 1 \) has a unique real root. Between which two whole numbers does this root lie? Justify your answer.

\[ f(x) \] computing the values of \( f(x) \) for several whole values of \( x \), we get

\[ f(-2) = -7 \]
\[ f(-1) = -1 \]
\[ f(0) = -1 \]
\[ f(1) = -1 \]
\[ f(2) = 5 \]

Thus, we see that \( f(x) \) changes sign on the interval \([1, 2] \). Since any polynomial is continuous, by Intermediate Value Theorem \( f(x) \) must have a root somewhere on this interval. Thus, the root is between 1 and 2.

6. It is known that for a rectangular beam of fixed length, its strength is proportional to \( w \cdot h^2 \), where \( w \) is the width and \( h \) is the height of the beam’s cross-section.

Find the dimensions of the strongest beam that can be cut from a 12” diameter log (thus, the cross-section must be a rectangle with diagonal 12”).

**Solution:** The dimensions of the beam are width \( w \) and height \( h \). They must satisfy the conditions \( h \geq 0, w \geq 0 \). In addition, since the diagonal of the cross-section must be 12 inches, Pythagorean theorem gives \( h^2 + w^2 = 12^2 = 144 \). Thus, we need to find the maximum of the function \( wh^2 \), where \( h, w \) are real numbers subject to the above conditions.

Let us rewrite everything in terms of \( w \). Then \( h = \sqrt{144 - w^2} \); restrictions \( h \geq 0, w \geq 0 \) give \( 0 \leq w \leq 12 \), and the strength is given by

\[ s(w) = w(\sqrt{144 - w^2})^2 = w(144 - w^2) = -w^3 + 144w \]

So we need to find the maximum of this function on the interval \([0, 12]\).

\[ f'(w) = -3w^2 + 144 \]

so critical points are when

\[ -3w^2 + 144 = 0 \]
\[ 144 = 3w^2 \]
\[ w^2 = 48 \]
\[ w = \pm\sqrt{48} = \pm\sqrt{16 \cdot 3} = \pm4\sqrt{3} \]

Thus, on \([0, 12]\) there is a unique critical point, \( w = 4\sqrt{3} \).

To find the maximum, we compare the values of the function at the critical point and the endpoints:

\[ f(0) = 0(144 - 0^2) = 0 \]
\[ f(12) = 12(144 - 12^2) = 0 \]
\[ f(4\sqrt{3}) = 4\sqrt{3}(144 - (4\sqrt{3})^2) = 4\sqrt{3}(144 - 48) = 4\sqrt{3} \cdot 96 \]
Clearly, the largest value is \( f(4\sqrt{3}) \); thus, this is the maximum. So the best width is \( 4\sqrt{3} \), and the corresponding height is \( h = \sqrt{144 - w^2} = \sqrt{96} = 4\sqrt{6} \).

7. The curve defined by the equation

\[
y^2(y^2 - 4) = x^2(x^2 - 5)
\]

is known as the “devil’s curve”. Use implicit differentiation to find the equation of the tangent line to the curve at the point \((0; -2)\).

Solution: Rewriting the equation in the form

\[
y^4 - 4y^2 = x^4 - 5x^2
\]

and taking derivative of both sides, we get \( y'(4y^3 - 8y) = 4x^3 - 10x \), so

\[
y' = \frac{4x^3 - 10x}{4y^3 - 8y}
\]

Substituting \( x = 0, y = -2 \), we get \( y' = 0 \), so the tangent line is horizontal and the equation of the tangent line is \( y = -2 \).

8. Find the most general function \( f(x) \) satisfying

(a) \( f''(x) = \cos x \)

Solution: We have \( f'(x) = \sin x + C_1 \), so that

\[
f(x) = -\cos x + C_1x + C_2,
\]

where \( C_1 \) and \( C_2 \) are arbitrary constants.

(b) \( f'(x) = \frac{x^2 + x + 1}{x} \)

Solution: Since \( f'(x) = x + 1 + \frac{1}{x} \),

\[
f(x) = \frac{x^2}{2} + x + \ln |x| + C,
\]

where \( C \) is an arbitrary constant.