Practice Midterm 1 Solutions

MAT 125, Spring 2008

Name:	**ID #:**	**Section:**

Please answer each question in the space provided. Show your work whenever possible. Unless otherwise marked, **answers without justification will get little or no partial credit**. Cross out anything the grader should ignore and circle or box the final answer.

(1) If $f(x) = \ln x$ and $g(x) = x^2 - 4$, find the functions $f \circ f$, $f \circ g$, $g \circ f$, $g \circ g$, and their domains.

Solution:

(a) $(f \circ f)(x) = \ln(\ln x)$, and domain consists of such x that both inequalities, $x > 0$ and $\ln x > 0$ are satisfied. The second inequality, by exponentiating, gives $x = e^{\ln x} > e^0 = 1$, so that domain is $(1, \infty)$.

(b) $(f \circ g)(x) = \ln(x^2 - 4)$, and domain consists of such x that $x^2 - 4 > 0$. Solving this inequality gives $x < -2$ and $x > 2$, so that the domain is $(-\infty, 2) \cup (2, \infty)$.

(c) $(g \circ f)(x) = (\ln x)^2 - 4$, and domain is $(0, \infty)$.

(d) $(g \circ g)(x) = (x^2 - 4)^2 - 4$, and domain is $\mathbb{R}$.

(2) Calculate the following limits

(a) $\lim_{x\to 2} 3x^2 + x - 2$

Solution: Since $f(x) = 3x^2 + x - 2$ is continuous, $\lim_{x\to 2} f(x) = f(2) = 12$.

(b) $\lim_{y\to -3} |y + 3|$

Solution: For $y > -3$, $y + 3 > 0$ so $|y + 3| = y + 3$. Thus, $\lim_{y\to(-3)+} |y+3| = \lim_{y\to(-3)+} y+3 = (-3)+3 = 0$. Similarly, $\lim_{y\to(-3)-} |y + 3| = \lim_{y\to(-3)-} -(y + 3) = -((-3) + 3) = 0$. Since one-sided limits coincide, $\lim_{y\to -3} |y + 3| = 0$.

(c) $\lim_{x\to 2} \frac{x^2+x-6}{x-2}$

Solution: In this case, plugging in $x = 2$ is impossible because in this case both the numerator and denominator are zero. Instead, we can factor the numerator, using the formula for roots of quadratic equation, to get

$$\lim_{x\to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x\to 2} \frac{(x + 3)(x - 2)}{x - 2}$$
$$= \lim_{x\to 2}(x + 3) = 5$$

(d) $\lim_{q\to 2} \frac{2q^2+5}{\sqrt{q+2}}$

Solution: This function is continuous, thus $\lim_{q\to 2} f(q) = f(2) = (2 \cdot 4 + 5)/\sqrt{4} = 13/2 = 6.5$

(e) $\lim_{t\to 3} \frac{\sqrt{t}-\sqrt{3}}{t-3}$

Solution: Again, both numerator and denominator have limit zero, so we can not use the quotient rule; instead, we can multiply both numerator and denominator by $\sqrt{t}+\sqrt{3}$:

$$\lim_{t\to 3}\frac{\sqrt{t}-\sqrt{3}}{t-3}=\lim_{t\to 3}\frac{(\sqrt{t}-\sqrt{3})(\sqrt{t}+\sqrt{3})}{(t-3)(\sqrt{t}+\sqrt{3})}$$
$$=\lim_{t\to 3}\frac{t-3}{(t-3)(\sqrt{t}+\sqrt{3})}=\lim_{t\to 3}\frac{1}{\sqrt{t}+\sqrt{3}}=\frac{1}{2\sqrt{3}}$$

(f) $\lim_{s\to 0} s^2\cos\left(s+\frac{1}{s}\right)$

Solution: Denote $f(s)=s^2\cos\left(s+\frac{1}{s}\right)$.
Since $-1\le\cos\left(s+\frac{1}{s}\right)\le 1$, we have $-s^2\le f(s)\le s^2$. Since $\lim_{s\to 0}s^2=\lim_{s\to 0}(-s^2)=0$, by squeeze theorem we have $\lim_{s\to 0}f(s)=0$.

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(3) Calculate

$$\lim_{x\to 1}\left(\frac{1}{x-1}+\frac{1}{x^2-3x+2}\right)$$

Solution: We can not plug-in $x=1$, since both fractions are not defined at $x=1$. However, factoring the polynomial x^2-3x+2 as

$$x^2-3x+2=(x-1)(x-2)$$

allows to simplify the sum of these two fractions for $x\neq 1$

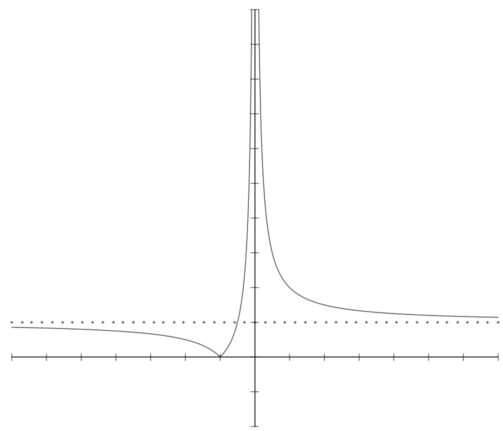
$$\frac{1}{x-1}+\frac{1}{x^2-3x+2}=\frac{1}{x-1}+\frac{1}{(x-1)(x-2)}+\frac{x-2+1}{(x-1)(x-2)}=\frac{1}{x-2}.$$

Thus

$$\lim_{x\to 1}\left(\frac{1}{x-1}+\frac{1}{x^2-3x+2}\right)=\lim_{x\to 1}\frac{1}{x-2}=\frac{1}{1-2}=-1.$$

(4) Let $f(x)=\left|1+\frac{1}{x}\right|$.
(a) Sketch the graph of f.
(b) Find all values of x for which f is not continuous.

Solution: The graph is shown below; it is obtained from the graph of $y=\frac{1}{x}$ by shifting it one unit up (this gives graph of $y=1+\frac{1}{x}$) and then reflecting the part of the graph below the x-axis.



Since the functions $1/x$ and $|x|$ are continuous, $f(x)$ is also continuous. Thus, the only discontinuity points are when the function is not defined, that is, at $x = 0$.

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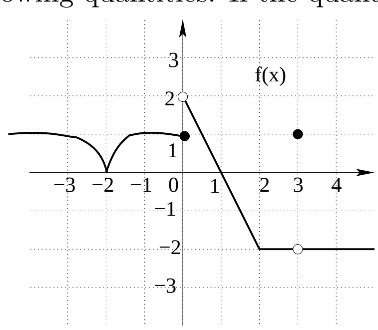
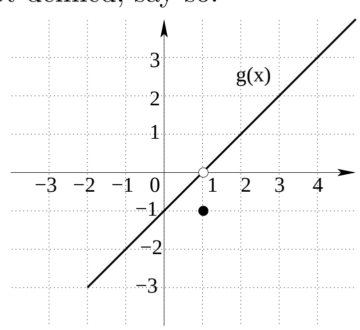
(5) Use the graphs of $f(x)$ and $g(x)$ below to compute each of the following quantities. If the quantity is not defined, say so.





$f(0)$ $\qquad \lim\limits_{x\to 0+} f(x)$ $\qquad \lim\limits_{x\to 0-} f(x)$ $\qquad \lim\limits_{x\to 0} f(x)$

$\lim\limits_{x\to 1} g(x)$ $\qquad \lim\limits_{x\to 1} f(x) - g(x)$ $\qquad \lim\limits_{x\to 3}(2f(x) - f(3))$

Solution: $f(0) = 1$; $\lim_{x\to 0+} f(x) = 2$; $\lim_{x\to 0-} f(x) = 1$; $\lim_{x\to 0} f(x)$ does not exist, since the one-sided limits are different; $\lim_{x\to 1} g(x) = 0$;
$\lim_{x\to 1} f(x) - g(x) = \lim_{x\to 1} f(x) - \lim_{x\to 1} g(x) = 0 - 0 = 0$;
$\lim_{x\to 3}(2f(x) - f(3)) = \left(2\lim_{x\to 3} f(x)\right) - f(3) = 2(-2) - 1 = -5$.

(6) Consider the function

$$f(t) = \begin{cases} \frac{t}{t-1} & t \geq 0 \\ t+1 & t < 0 \end{cases}$$

(a) At which points is this function continuous?
(b) Find the left and right limits, if they exist, at $t = 0$.

Solution:

For $t < 0$, this function is given by $f(t) = t + 1$, so it is continuos.

For $t > 0$, this function is given by $f(t) = \frac{t}{t-1}$, so it is continuos wherever defined. Thus, it is continuous at all points where denominator is non-zero, i.e. $t \neq 1$

It remians to consider the point $t = 0$. At this point, function is defined by different formulas on two sides of this point. To check whether it is continuous, we compute the one-sided limits.

$$\lim_{t \to 0+} f(t) = \lim_{t \to 0+} \frac{t}{t-1} = \frac{0}{0-1} = 0$$
$$\lim_{t \to 0-} f(t) = \lim_{t \to 0-} (t+1) = 0 + 1 = 1$$

Since these limtis are not equal, limit $\lim_{t \to 0} f(t)$ does not exist. So $f(t)$ is not continuous at 0.

Thus, $f(t)$ is continuous everywhere except $t = 0$, $t = 1$.