Section 8.4

2.

a) \[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 8 \]
   \[ \sum a_n \] is divergent.

b) \[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 0.8 \]
   \[ \sum a_n \] is convergent.

c) \[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1 \]
   \[ \sum a_n \] may or may not converge, that is, we cannot determine that the series is convergent or divergent using this test.

4.

\[ -\frac{1}{3} + \frac{2}{4} - \frac{3}{5} + \frac{4}{6} - \frac{5}{7} + \cdots \]

Let the nth term of the series is \( a_n \) Then

\[ a_n = (-1)^n \frac{n}{n+2} \]

Thus \[ \lim_{n \to \infty} a_n \] does not exist.

Therefore, \[ \sum a_n \] does not converge.

12.

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \] (error \( < 0.001 \))

Let \( b_n = \frac{1}{n^4} \), then \( b_n < b_{n+1} \) and \[ \lim_{n \to \infty} b_n = 0. \]

Hence, the series is convergent.

Moreover, \[ |R_n| \leq b_{n+1} \]

Then if \[ \frac{1}{n^4} < 0.001 \], then \[ n > \frac{1}{\sqrt[4]{0.001}} \approx 5.623413 \]

Therefore, we need 5 terms.

22.
\[ \sum_{n=0}^{\infty} \frac{(-3)^n}{n!} \quad \text{Let } a_n = \frac{(-3)^n}{n!} \]

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \lim_{n \to \infty} \frac{3}{n+1} = 0 \]

Therefore, \( \sum a_n \) is absolutely convergent.

24.

\[ \sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2 + 1} \quad \text{Let } a_n = (-1)^n \frac{n}{n^2 + 1} \]

\[ |a_n| = \frac{n}{n^2 + 1} \geq \frac{n}{2n^2} = \frac{1}{2n}, \quad \text{for all } n \geq 1 \]

\[ \sum_{n=1}^{\infty} \frac{1}{2n} \text{ is divergent, so } \sum a_n \text{ is also divergent by the Comparison Test.} \]

Therefore, the series does not converge absolutely. (However, the series is convergent by the test for alternating series.)

30.

\[ a_1 = 1, \quad a_{n+1} = \frac{2 + \cos n}{\sqrt{n}} \cdot a_n \]

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{2 + \cos n}{\sqrt{n}} \leq \lim_{n \to \infty} \frac{3}{\sqrt{n}} = 0, \quad \text{So that } \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 \]

Therefore, the series is absolutely convergent.

34.

\[ \sum a_n, \quad r_n = \frac{a_{n+1}}{a_n}, \quad \text{and } \lim_{n \to \infty} r_n = L < 1 \]

\[ R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \]

a) Prove that if \( r_n > r_{n+1} \), and \( r_{n+1} < 1 \), then \( R_n \leq \frac{a_{n+1}}{1 - r_{n+1}} \)
\[ R_n = a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4} + \cdots \]
\[ = a_{n+1} + \frac{a_{n+2}}{a_{n+1}} \cdot a_{n+2} + \frac{a_{n+3}}{a_{n+2}} \cdot a_{n+3} + \frac{a_{n+4}}{a_{n+3}} \cdot a_{n+4} + \cdots \]
\[ = a_{n+1} \left( 1 + \frac{a_{n+2}}{a_{n+1}} \cdot \frac{a_{n+3}}{a_{n+2}} \cdot \frac{a_{n+4}}{a_{n+3}} \cdot \cdots \right) \]
\[ = a_{n+1} \left( 1 + r_{n+1} + r_{n+2} \cdot r_{n+1} + r_{n+3} \cdot r_{n+2} \cdot r_{n+1} + \cdots \right) \]
\[ \leq a_{n+1} \left( 1 + r_{n+1} + r_{n+2} + r_{n+3} + \cdots \right) \]
\[ = a_{n+1} \cdot \frac{1}{1 - r_{n+1}} \]

Remark: \(|r_{n+1}| < 1\), and \(\{r_{n+1}\}\) is decreasing, so \((*)\) is convergent.

Therefore, \(R_n \leq \frac{a_{n+1}}{1 - r_{n+1}}\)

b) Prove that if \(\{r_n\}\) is the increasing sequence, then \(R_n \leq \frac{a_{n+1}}{1 - L}\)

\[ R_n = a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4} + \cdots \]
\[ = a_{n+1} + \frac{a_{n+2}}{a_{n+1}} \cdot a_{n+2} + \frac{a_{n+3}}{a_{n+2}} \cdot a_{n+3} + \frac{a_{n+4}}{a_{n+3}} \cdot a_{n+4} + \cdots \]
\[ = a_{n+1} \left( 1 + \frac{a_{n+2}}{a_{n+1}} \cdot \frac{a_{n+3}}{a_{n+2}} \cdot \frac{a_{n+4}}{a_{n+3}} \cdot \cdots \right) \]
\[ = a_{n+1} \left( 1 + r_{n+1} + r_{n+2} \cdot r_{n+1} + r_{n+3} \cdot r_{n+2} \cdot r_{n+1} + \cdots \right) \]
\[ \leq a_{n+1} \left( 1 + L + L^2 + L^3 + \cdots \right) \]
\[ = a_{n+1} \cdot \frac{1}{1 - L} \]

Remark: \(\{r_n\}\) is increasing sequence, so that \(r_n \leq L\) for all \(n\).

Therefore, \(R_n \leq \frac{a_{n+1}}{1 - L}\)