8.3

2.
x ≥ 1, \( f(n) = a_n \), and \( f(x) \) is a decreasing function.

\( a_n \) can be regraded as the area of rectangle whose length of height is \( a_n \) and that of base is 1.

In that case, \( \sum_{n=1}^{5} a_n \) is the upper sum and \( \sum_{n=2}^{6} a_n \) is the lower sum for \( 1 \leq x \leq 6 \).

Therefore, \( \sum_{n=2}^{6} a_n < \int_{1}^{6} f(x) < \sum_{n=1}^{5} a_n \). This inequality is dependent on the fact that \( f(x) \) is a decreasing function.

Draw the graph of \( f(x) \) and upper sum and lower sum for \( 1 < x < 6 \).

12.

\[ \sum_{n=1}^{\infty} \left( \frac{5}{n^4} + \frac{4}{n\sqrt{n}} \right) \]

Consider \( \sum_{n=1}^{\infty} \frac{5}{n^4} \) \( \cdots \) \( (*) \) and \( \sum_{n=1}^{\infty} \frac{4}{n\sqrt{n}} \) \( \cdots \) \( (**) \)

Use the integral test.

\[ \int_{1}^{\infty} \frac{5}{x^4} = \left[ -\frac{5}{3x^3} \right]_{1}^{\infty} = \frac{5}{3} < \infty \quad \text{and} \quad \int_{1}^{\infty} \frac{4}{x\sqrt{x}} = \left[ -\frac{8}{\sqrt{x}} \right]_{1}^{\infty} = 8 < \infty \]

Hence, both \( (*) \) and \( (**) \) are convergent, so the given series is also convergent.

14.

\[ \sum_{n=1}^{\infty} \frac{\ln n}{n^2} \]

Use the integral test.

\[ \ln x = t \Rightarrow x = e^t \]

\[ \frac{1}{x} \frac{dx}{dt} = \frac{dx}{e^t} = \frac{1}{x} \frac{t}{e^t} dt = \frac{1}{x} e^{-t} dt = \int_{0}^{\infty} e^{-t} dt = \left[ -(t+1)e^{-t} \right]_{0}^{\infty} = 1 < \infty \]

because \( \lim_{t \to \infty} \left( -\frac{t+1}{e^t} \right) = 0 \) (Use the L’Hospital’s law)

Therefore, the given series is convergent.

18.
\[
\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^{\sqrt{n}}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{n}}} \quad \text{because} \quad \sin^2 n \leq 1
\]

Use the integral test.
\[
\int_{1}^{\infty} \frac{1}{x^{\sqrt{x}}} dx = \left[-\frac{2}{\sqrt{x}}\right]_{1}^{\infty} = 2 < \infty
\]

So \(\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{n}}}\) is convergent.

Therefore, the given series is also convergent (by Comparison Test).

20.
\[
\sum_{n=1}^{\infty} \frac{4 + 3^n}{2^n}
\]
\[
\lim_{n \to \infty} \frac{4 + 3^n}{2^n} \geq \lim_{n \to \infty} \frac{3^n}{2^n} = \lim_{n \to \infty} \left(\frac{3}{2}\right)^n = \infty \neq 0
\]

Therefore, the given series is divergent.

30.
\[
\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}
\]
\[
\ln x = t \Rightarrow x = e^t \quad x = \infty \Rightarrow t = \infty
\]
\[
\int_{2}^{\infty} \frac{1}{x(\ln x)^2} dx = \int_{\ln 2}^{\infty} \frac{1}{t^2} dt = \left[\frac{-1}{t}\right]_{\ln 2}^{\infty} = \ln 2 < \infty
\]

So that the given series is convergent.

Moreover,
\[
R_n \leq \int_{n}^{\infty} \frac{1}{x(\ln x)^2} dx = \left[\frac{-1}{t}\right]_{\ln n}^{\infty} = \frac{1}{\ln n} < 0.01 \quad \text{where} \quad x = e^t, \text{so} \quad x = n \Rightarrow t = \ln n
\]

Hence, \(n > e^{100} \approx 2.68812 \times 10^{43}\)

Therefore, we need (at least) the \(e^{100}\) terms to ensure accuracy to within 0.01.