8.3

2.

$x \geq 1,\ f(n) = a_n,\ and\ f(x)$ is a decreasing function.

a_n can be regraded as the area of retangle whose length of height is a_n and that of base is1.

In that case, $\sum_{n=1}^{5} a_n$ *is the upper sum and* $\sum_{n=2}^{6} a_n$ *is the lower sum for* $1 \leq x \leq 6$.

Therefore, $\sum_{n=2}^{6} a_n < \int_1^6 f(x) < \sum_{n=1}^{5} a_n$. *This inequality is dependent on the fact that* $f(x)$ *is a decreasing fuction.*

Draw the graph of f(x) and upper sum and lower sum for $1 < x < 6$.

12.

$$\sum_{n=1}^{\infty}\left(\frac{5}{n^4} + \frac{4}{n\sqrt{n}}\right)$$

Consider $\sum_{n=1}^{\infty}\frac{5}{n^4}$ ----(*) *and* $\sum_{n=1}^{\infty}\frac{4}{n\sqrt{n}}$ ---(**)

Use the integral test.

$$\int_1^{\infty}\frac{5}{x^4} = \left[-\frac{5}{3x^3}\right]_1^{\infty} = \frac{5}{3} < \infty \quad and \quad \int_1^{\infty}\frac{4}{x\sqrt{x}} = \left[-\frac{8}{\sqrt{x}}\right]_1^{\infty} = 8 < \infty$$

Hence, both (*) *and* (**) *are convergent, so the given series is also convergent.*

14.

$\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ Use the integral test.

$\int_1^{\infty}\frac{\ln x}{x^2}dx$

$\ln x = t \Rightarrow x = e^t$

$\frac{1}{x}dx = dt$

$dx = e^t dt$

$x = 1 \Rightarrow t = 0$

$x = \infty \Rightarrow t = \infty$

$$= \int_0^{\infty}\frac{t}{e^{2t}}e^t dt = \int_0^{\infty}\frac{t}{e^t}dt = \int_0^{\infty}te^{-t}dt = \left[-(t+1)e^{-t}\right]_0^{\infty} = 1 < \infty$$

because $\lim_{t\to\infty}\left(-\frac{t+1}{e^t}\right) = 0$ (*Use the L'Hospital's law*)

Therefore, the given series is convergent.

18.

$$\sum_{n=1}^{\infty}\frac{\sin^2 n}{n\sqrt{n}} \le \sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}} \quad \textit{because} \quad \sin^2 n \le 1$$

Use the integral test.

$$\int_1^{\infty}\frac{1}{x\sqrt{x}} = \left[-\frac{2}{\sqrt{x}}\right]_1^{\infty} = 2 < \infty$$

So $\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}$ is convergent.

Therefore, the given series is also convergent (by Comparision Test).

20.

$$\sum_{n=1}^{\infty}\frac{4+3^n}{2^n}$$

$$\lim_{n\to\infty}\frac{4+3^n}{2^n} \ge \lim_{n\to\infty}\frac{3^n}{2^n} = \lim_{n\to\infty}\left(\frac{3}{2}\right)^n = \infty \ne 0$$

Therefore, the given series is divergent.

30.

$$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$$

$$\int_2^{\infty}\frac{1}{x(\ln x)^2}dx$$

$\ln x = t \Rightarrow x = e^t$

$\frac{1}{x}dx = dt$

$dx = xdt = e^t dt$

$x = \infty \Rightarrow t = \infty$

$x = 2 \Rightarrow t = \ln 2$

$$= \int_{\ln 2}^{\infty}\frac{1}{e^t t^2}e^t dt = \int_{\ln 2}^{\infty}\frac{1}{t^2}dt = \left[-\frac{1}{t}\right]_{\ln 2}^{\infty} = \ln 2 < \infty$$

So that the given series is convergent.

Moreover,

$$R_n \le \int_n^{\infty}\frac{1}{x(\ln x)^2}dx = \left[-\frac{1}{t}\right]_{\ln n}^{\infty} = \frac{1}{\ln n} < 0.01 \quad \textit{where} \;\; x = e^t, \textit{so} \;\; x = n \Rightarrow t = \ln n$$

Hence, $n > e^{100} \approx 2.68812\times10^{43}$

Therefore, we need (at least) the e^{100} terms to ensure accuracy to within 0.01.