In problems (1)-(4) below we consider a cylindrical rod centered along the x-axis in 3-dimensional space, from \( x = 0 \) to \( x = a \); for each \( 0 \leq x \leq a \) the intersection of this rod with the plane containing \((x,0,0)\) and perpendicular to the x-axis is a disc \( D_x \) of area \( A \). We assume that the physical properties of this rod are the same at each of its points: in particular its density function is a constant function \( \rho \), and the heat capacity per unit mass for the rod is also a constant function \( c \) (see page 36). We assume that for each time \( t \geq 0 \) and for each \( 0 \leq x \leq a \) the temperature at each point of \( D_x \) is equal to the same value \( u(x,t) \). Finally we assume that the cylindrical surface of the rod is insulated (the ends of the rod are not necessarily insulated).

(1) What does the heat flux function \( q(x,t) \) measure? State Fourier’s law of heat conduction for this rod.

**Solution:** See bottom of page 135 and top middle of page 137.

(2) Suppose that the rod is also insulated at its right hand end \( D_a \) and is kept at a constant temperature of 2 degrees celcius at its left hand end \( D_0 \).

(a) Give a mathematical description (some equations) of all these conditions placed on \( u(x,t) \), \( 0 \leq x \leq a \) and \( 0 \leq t \).

**Solution:**

\[
\frac{\partial^2 u}{\partial x^2} = \frac{1}{k} \frac{\partial u}{\partial t}
\]

\[
u(0,t) = 2, \quad \frac{\partial u}{\partial x}(a,t) = 0
\]

(b) Find a general solution to the equations in part (a).

**Solution:**

\[
u(x,t) = 2 + \sum_{n=1}^{\infty} b_n \sin(\lambda_n x) e^{-\lambda_n^2 k t}
\]

where \( \lambda_n = \frac{(2n-1)\pi}{2a} \).

(3) Let \( H(x,t) \) denote the total heat contained within the portion of the rod between \( D_0 \) and \( D_x \). Recall that \( H(x,t) = \int_0^x \rho c A u(y,t) dy \) (see page 136 of text).

Suppose that the rod is insulated at its right hand end \( D_a \) and that \( H(a,2) < H(a,0) \). Then show that \( \frac{\partial H}{\partial x}(0,t_0) > 0 \) holds for some \( 0 \leq t_0 \leq 2 \).

**Solution:** The rod can loose heat only thru the disc \( D_0 \). It must loose heat thru the disc \( D_0 \) at some time \( 0 \leq t_o \leq 2 \) because \( H(a,2) < H(a,0) \). Thus the rate of heat flow thru \( D_0 \) at time \( t_o \) — which is equal to \( q(0,t_0)A \) — must be negative; i.e. \( q(0,t_0) < 0 \). Since \( q(x,t) = -k \frac{\partial u}{\partial x}(x,t) \) (why?), it follows that \( \frac{\partial H}{\partial x}(0,t_0) > 0 \).
(4) Suppose that \( u(x, t) \) satisfies \( u(x, 0) = 3 \) in addition to the properties of problem (2)(a) above.

(a) Compute \( H(a, 0) \) and \( \lim_{t \to \infty} H(a, t) \).

**Solution:** \( H(a, 0) = 3a \rho c A \).

\( \lim_{t \to \infty} H(a, t) \) should equal to the heat content of the bar for the steady state solution \( v(x) \). Note that \( v(x) = 2 \); so the heat content for the steady state solution is \( 2a \rho c A \).

(b) Verify that \( \frac{\partial u}{\partial x}(0, t) > 0 \) for all \( t > 0 \). (Hint: Write \( u(x, t) \) as an infinite series and compute its x-derivative term by term.)

**Solution:** \( u(x, t) \) is the solution to the equations of (2)(a) and the initial condition \( u(x, 0) = 3 \). Thus \( u(x, t) \) is equal to the infinite series of given in (2)(b), where the \( b_n \) in (2)(a) are given by \( b_n = \frac{2}{\pi} \int_0^\pi \sin(\lambda_n x) dx \). Thus \( \frac{\partial u}{\partial x}(0, t) = \sum_{n=1}^\infty \frac{2}{\pi} e^{-\lambda_n^2 k t}, \) which is clearly positive for all \( t > 0 \).

(c) Use part (b) to verify that \( H(a, t) \) is a decreasing function in \( t \).

**Solution:** Using Fourier’s Law, and part (b) of this problem, we conclude that \( q(x, t) < 0 \) for all \( t \). Thus heat is flowing out of the rod at \( D_0 \) for all \( t > 0 \); implying that the heat content of the rod \( H(a, t) \) is decreasing for all \( t > 0 \).

(5) For all \( 0 \leq x \leq \pi \) and \( 0 \leq t \) suppose that the following equations hold for the function \( u(x, t) \):

\[
(i) \quad \frac{\partial^2 u}{\partial x^2}(x, t) = \frac{\partial^2 u}{\partial t^2}(x, t) \\
(ii) \quad u(0, t) = 0, u(\pi, t) = 0 \\
(iii) \quad u(x, 0) = \sin(x), \frac{\partial u}{\partial t}(x, 0) = \sin(x)
\]

(a) find the d’Alembert solution to these equations.

**Solution:** \( u(x, t) = \frac{\sin(x+t) + \sin(x-t)}{2} + \frac{\cos(x-t) - \cos(x+t)}{2} \)

(b) find the Fourier type solution to these equations.

**Solution:** \( u(x, t) = \sum_{n=1}^{\infty} a_n \sin(nx)(\cos(nt) + \sin(nt)), \) where \( a_n = \frac{2}{\pi} \int_0^\pi \sin(x) \sin(nx) dx \) and \( b_n = \frac{2}{\pi} \int_0^\pi \sin(x) \sin(nx) dx \). Thus \( u(x, t) = \sin(x)(\cos(t) + \sin(t)). \)

(c) does this vibrating string ever return to its original position?

**Solution:** Using the solution in (b) above, we see that \( u(x, t) \) is periodic of period \( 2\pi \) in the \( t \) variable. So the string returns to its original position after \( 2\pi \) amount of time has elapsed.

(6) Show that if \( u_1(x, t) \) and \( u_2(x, t) \) both satisfy equations (i),(ii) in problem (5), then \( u(x, t) = \alpha_1 u_1(x, t) + \alpha_2 u_2(x, t) \) also satisfies (i),(ii) in problem (5) for any real numbers \( \alpha_1, \alpha_2 \).

**Solution:** This uses the homogeneity of (i)(ii).
To verify (i):
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \sum_{i=1}^{2} \alpha_i \left( \frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial y^2} \right) = \sum_{i=1}^{2} \alpha_i \frac{1}{c^2} \frac{\partial^2 u_i}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}.
\]

To verify (ii): \( u(0, t) = \sum_{i=1}^{2} \alpha_i u_i(0, t) = \sum_{i=1}^{2} \alpha_i 0 = 0; \) \( u(a, t) = \sum_{i=1}^{2} \alpha_i u_i(a, t) = \sum_{i=1}^{2} \alpha_i 0 = 0. \)

(7) Do problem (11) on page 232 of the text.

(8) Suppose that \( u(x, t), 0 \leq x, t, \) satisfies
\[
\frac{\partial u}{\partial x}(x, t) = -\frac{1}{c} \frac{\partial u}{\partial t}(x, t)
\]
\[
\begin{align*}
 u(0, t) &= 0 \\
 u(x, 0) &= f(x)
\end{align*}
\]
for some given differentiable function \( f(x) \).

(a) Show that \( u(x, t) \) also satisfies
\[
\frac{\partial^2 u}{\partial x^2}(x, t) = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}(x, t)
\]

and
\[
\frac{\partial u}{\partial t}(x, 0) = -cf'(x).
\]

**Solution:** The first equality is derived as follows:
\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{-1}{c} \frac{\partial u}{\partial t} \right) = -\frac{1}{c} \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} \right) = -\frac{1}{c} \frac{\partial}{\partial t} \left( \frac{-1}{c} \frac{\partial u}{\partial t} \right) = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}
\]

The second equality is derived as follows:
\[
\frac{\partial u}{\partial t}(x, 0) = -\frac{\partial u}{\partial x}(x, 0) = -cf'(x).
\]

(b) Solve for \( u(x, t) \) in terms of the function \( f \).

**Solution:** In part (a) we showed that \( u \) satisfies the wave equation and has initial position \( u(x, 0) = f(x) \) and initial velocity \( \frac{\partial u}{\partial t}(x, 0) = g(x) = -cf'(x) \). Thus — by d’Alembert — we have that
\[
 u(x, t) = \frac{1}{2} (f(x + ct) + f(x - ct) + G(x + ct) - G(x - ct))
\]
where
\[
G(x) = \frac{1}{c} \int_{0}^{x} g(s)ds = \frac{1}{c} \int_{0}^{x} -cf'(s)ds = -\int_{0}^{x} f'(s)ds = -(f(x) - f(0)).
\]

Combining these last two equalities we get that
\[
 u(x, t) = \frac{1}{2} (f_0(x - ct) + f_0(x + ct)).
\]
Note that \( \frac{1}{2} (f_0 + f_c) = \hat{f} \) — where \( \hat{f}(x) = f(x) \) if \( x \geq 0 \) and \( \hat{f}(x) = 0 \) if \( x < 0 \).

(c) Give a physical description of the solution of part (b).

**Solution:** This is a traveling wave, moving from left to right.

(9) A real valued function \( u(x, y) \) of the two real variables \( x, y \) is harmonic if it satisfies
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0
\]
on its domain.

(a) If \( u(x, y) = \sum_{0 \leq i+j \leq 3} a_{i,j} x^i y^j \), and \( u \) is harmonic in a disc of radius 2 centered at \((-3,4)\), then prove that \( u \) is harmonic on the whole plane.

**Solution:** \( u \) is harmonic on the disc iff \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \) holds on the disc.

Note that \( \frac{\partial^2 u}{\partial x^2} = 2a_{2,1} + 6a_{3,0} \) and \( \frac{\partial^2 u}{\partial y^2} = 2a_{1,2} + 5a_{0,3} \). Thus we have that \( u \) is harmonic on the disc iff
\[
2a_{2,1} + 5a_{0,3} = 0
\]
and
\[
6a_{3,0} + 2a_{1,2} = 0.
\]
Note that these last two equalities also equivalent to \( u \) being harmonic on the whole plane.

(b) It is a fact that if \( u \) is harmonic on a finite rectangle \( \mathbb{R} = \{(x, y) | a \leq x \leq b, c \leq y \leq d\} \), then it takes on neither a maximum value nor a minimum value in the interior of this rectangle \( \{(x, y) | a < x < b, c < y < d\} \). Prove this fact under the additional hypothesis that \( \frac{\partial^2 u}{\partial x^2} \) does not vanish in the interior of the rectangle.

**Solution:** If \( u \) takes on a maximum or minimum at a point \( p \) inside of \( \mathbb{R} \), then \( p \) must be a critical point for \( u \). Now apply the second derivative test for \( u \) at \( p \); you will see (using the hypothesis for \( u \)) that \( p \) is a saddle point for \( u \).

(10) Consider the following 2-dimensional heat problem:
\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{k} \frac{\partial u}{\partial t}
\]
\[
u(x, 0, t) = 0, \quad u(x, b, t) = 0
\]
\[
u(0, y, t) = 3 \sin \left( \frac{2\pi}{b} \right) y, \quad u(a, y, t) = - \sin \left( \frac{5\pi}{b} \right) y
\]
\[
u(x, y, 0) = x + y
\]
Find the steady state solution for this problem.
Solution: The steady state solution $v(x, y)$ satisfies the following equations:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$v(x, 0) = 0, v(x, b) = 0, v(0, y) = 3\sin\left(\frac{2\pi}{b} y\right), v(a, y) = -\sin\left(\frac{5\pi}{b} y\right).$$

So we may use the results of section 4.2 (with the roles of $x, a$ and $y, b$ reversed) to conclude that

$$v(x, y) = \sum_{n=1}^{\infty} (a_n e^{\lambda_n x} + b_n e^{\lambda_n x}) \sin(\lambda_n y)$$

where $\lambda_n = \frac{n\pi}{b}$. We can solve for $a_n, b_n$ by comparing the above form of $v(x, y)$ with the last two boundary conditions. Thus $3\sin(\lambda_2 y) = v(0, y) = \sum_{n=1}^{\infty} (a_n + b_n) \sin(\lambda_n y)$, which implies that

$$3 = a_2 + b_2$$

$$0 = a_n + b_n, \quad n \neq 2.$$

Also $-\sin(\lambda_5 y) = v(a, y) = \sum_{n=1}^{\infty} (a_n e^{\lambda_n a} + b_n e^{-\lambda_n a}) \sin(\lambda_n y)$, which implies

$$-1 = a_5 e^{\lambda_5 a} + b_5 e^{-\lambda_5 a}$$

$$0 = a_n e^{\lambda_n a} + b_n e^{-\lambda_n a}, \quad n \neq 5.$$

We can solve the preceding 4 displayed equalities for $a_n, b_n$: in particular $a_n = 0 = b_n$ if $n \neq 2, 5$. 