SOLUTIONS TO MIDTERM FOR MAT 341

Instructions: Do each of the following 4 problems in the spaces provided. Be sure to show some work or give an explanation for each of your answers. Also print your name and ID number in the spaces below.

A table of integral formulae can be found on the last page of this exam.

Print Name:

Write ID number:

Patchogue.math.sunysb.edu [ftex] 58) latex solmidter

(1) Consider the function $f(x) = 2x - 1, \quad -2 < x < 2.$

(a) (worth 8 points) Is $f(x)$ an even function? Is $f(x)$ and odd function?
Solution: $f(1) = 1, f(-1) = -3$; so $f(1) \neq f(-1)$ and $f(1) \neq -f(-1)$. Thus $f(x)$ is neither even nor odd.

(b) (worth 9 points) Compute the Fourier series for $f(x)$.
Solution: $g(x) = x$, $-2 < x < 2$, is an odd function, so its Fourier series is equal $g(x) \sim \sum_{n=1}^{\infty} b_n sin(\frac{n\pi}{2}x)$, where $b_n = \frac{2}{2}\int_0^2 x sin(\frac{n\pi}{2}x)dx = (4\frac{sin(\frac{n\pi}{2}x)}{n^2\pi^2} - 2\frac{xcos(\frac{n\pi}{2}x)}{n\pi}) \mid_0^2 = \frac{4(-1)^{n+1}}{n\pi}$. The Fourier series for $h(x) = 1, -2 < x < 2$, is $h(x) \sim 1$. Since $f(x) = 2g(x) - h(x)$ its Fourier series is

$$f(x) \sim -1 + \sum_{n=1}^{\infty} \frac{8(-1)^{n+1}}{n\pi} sin(\frac{n\pi}{2}x).$$

(c) (worth 8 points) At each $x \in [-6, 6]$ compute the value to which the Fourier series of part (b) above converges.
Solution: Note that the periodic extension of $f(x)$ to the whole real line — denoted by $h(x)$ — is sectionally smooth and is continuous except at $x = 4n + 2$ for $n = integer$. It follows (see Theorem page 76) that the Fourier series at x converges to $h(x)$ if $x \neq 4n + 2$, and converges to $\frac{h(x-)+h(x+)}{2}$ if $x = 4n + 2$. Thus the Fourier series converges to -1 at $x = -6, -2, 2, 6$; it converges to $2x - 1$ at any $x \in (-2, 2)$; it converges to $2(x + 4) - 1$ for any $x \in (-6, -2)$; and it converges to $2(x - 4) - 1$ at any $x \in (2, 6)$.

(2) (worth 25 points) Consider the following equations:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(a,t) = 0$$

where $u(x,t)$ denotes a function of the real variables x, t. Suppose that both of the functions $f(x,t), g(x,t)$ satisfy these equations (when substituted for $u(x,t)$ in the equations). Then prove that the function $h(x,t)$ — defined by $h(x,t) = 2f(x,t) - 3g(x,t)$ — also satisfies these equations (when $h(x,t)$ is substituted for $u(x,t)$ in the equations).

Solution: We have that both $\frac{\partial^2 f}{\partial x^2} = \frac{1}{k}\frac{\partial f}{\partial t}$ and $\frac{\partial^2 g}{\partial x^2} = \frac{1}{k}\frac{\partial g}{\partial t}$; thus

$$\frac{\partial^2 h}{\partial x^2} = \frac{\partial^2(2f-3g)}{\partial x^2} = 2\frac{\partial^2 f}{\partial x^2} - 3\frac{\partial^2 g}{\partial x^2} = 2\frac{\partial f}{\partial t} - 3\frac{\partial g}{\partial t} = \frac{\partial(2f-3g)}{\partial t} = \frac{\partial h}{\partial t}.$$

We also have that both $f(0,t) = 0 = \frac{\partial f}{\partial x}(a,t)$ and $g(0,t) = 0 = \frac{\partial g}{\partial x}(a,t)$; thus

$$h(0,t) = 2f(0,t) - 3g(0,t) = 0 = 2\frac{\partial f}{\partial x}(a,t) - 3\frac{\partial g}{\partial x}(a,t) = \frac{\partial h}{\partial x}(a,t).$$

(3) Find a function $u(x,t)$ which satisfies all the following equations:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\frac{\partial u}{\partial t}$$
$$u(0,t) = 1, \quad u(1,t) = -2$$
$$u(x,0) = -3x.$$

Solution: The steady state solution has the form $v(x) = \alpha x + \beta$. The first boundary condition (for $v(x)$) states $\alpha 0 + \beta = 1$; thus $\beta = 1$. The second boundary condition (for $v(x)$) states that $\alpha 1 + 1 = -2$; thus $\alpha = -3$. Thus

$$v(x) = -3x + 1.$$

Now we set $w(x,t) = u(x,t) - v(x)$. This function satisfies

$$(1) \quad \frac{\partial^2 w}{\partial x^2} = \frac{1}{2}\frac{\partial u}{\partial t}$$
$$(2) \quad w(0,t) = 0, w(1,t) = 0$$
$$(3) \quad w(x,0) = -1.$$

Then general solution for this heat equation of (1) above and homogeneous boundary conditions of (2) above has the form

$$w(x,t) = \sum_{n=1}^{\infty} b_n sin(n\pi x)e^{-2n^2\pi^2 t}.$$

The initial condition for w(x,t) of (3) above is satisfied if the b_n are choosen as follows

$$b_n = 2\int_0^1 (-1)sin(n\pi x)dx = \frac{2}{n\pi}cos(n\pi x)\mid_0^1 = \frac{2}{n\pi}((-1)^n - 1).$$

Thus

$$w(x,t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}((-1)^n - 1)sin(n\pi x)e^{-2n^2\pi^2 t}$$

and

$$u(x,t) = w(x,t) + (-3x + 1).$$

(4) In parts (a) and (b) give a physical situation which the equations describe, and either find the steady state solution for the equations or give a physical explaination of why no steady steady state solution exists.

(a) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$, $u(0,t) = 1, -\kappa\frac{\partial u}{\partial x}(a,t) = (u(a,t) - T_1)h$, where k, κ, h, T_1 are positive constants.

Solution: These equations describe the heat distribution in a rod of length 1, which is insulated on the rounded part of its surface, kept a constant temperature 1 at its left hand end, and whose right hand end is exposed to a gas or liquid of constant temperature T_1 is subject (at its right hand end) to heat transference by convection.

The steady state solution has the form $v(x) = \alpha x + \beta$. The first boundary condition (for $v(x)$) states that $\alpha 0 + \beta = 1$; thus $\beta = 1$. Note that $\frac{dv}{dx} = \alpha$. Thus the second boundary condition (for $v(x)$) states that $-\kappa\alpha = (\alpha a + 1 - T_1)h$; from which we can solve for $\alpha = \frac{(T_1-1)h}{\kappa + ah}$. Thus

$$v(x) = (\frac{(T_1 - 1)h}{\kappa + ah})x + 1.$$

(b) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}(0,t) = -2, \frac{\partial u}{\partial x}(a,t) = 0$, where k is a positive constant.

Solution: These equations describe the the temperature distribution in a rod of length a which is insulated on the rounded part of its surface and at its right hand end, and experiences a heat flow in thru its left hand end at a positive constant rate. Thus the heat in the rod is always increasing with respect to time as is the temperature, i.e. $lim_{t\to\infty} u(x,t) = \infty$. So there is no steady state solution.