Sec. 2.7 4

The solution is of the form

$$\phi(x) = a\cos(\lambda x) + b\sin(\lambda x)$$

Therefore

$$\phi'(x) = -a\sin(\lambda x) + b\cos(\lambda x)$$

According to the boundary condition:

$$\alpha_1\phi(0) - \alpha_2\phi'(0) = \alpha_1 a - \alpha_2 b\lambda = 0$$

So up to a constant, the eigenfunction looks like the one given in the problem.

Take the derivative of the eigenfunction and consider the second boundary condition

$$0 = \beta_1(\alpha_2\lambda_n\cos(\lambda_n a) + \alpha_1\sin(\lambda_n a)) + \beta_2(-\alpha_2\lambda_n^2\sin(\lambda_n a) + \alpha_1\lambda_n\cos(\lambda_n a))$$

Then it is straightforward to get the fomula in the problem.

Sec 2.8 4

Since we are assuming the Sturm-Liouville problem is given in the form of sec 2.7, we know that they satisfy the orthogonal conditions. And the normalization condition is automatic because of the choice of the normalization factor.

Sec 2.10 7

If we want the method to work, we have to be able to integrate

$$B(\lambda) = \frac{2}{\pi}\int_0^\infty (f(x) - T_0)\sin(\lambda x)dx$$

Now let

$$\lim_{n\to\infty} f(x) = T_0 + A$$

then for any $\epsilon > 0$, there exists M large enough so that $\mid f(x) - T_0 - A \mid \leq \epsilon$ for any $x \geq M$.

So

$$\frac{2(A-\epsilon)}{\lambda} \leq \int_{\frac{2N\pi}{\lambda}}^{\frac{2N\pi}{\lambda}+\frac{\pi}{\lambda}} (f(x) - T_0)\sin(\lambda x)dx \leq \frac{2(A+\epsilon)}{\lambda}$$

for N large enough.

Now if A is not zero, we can choose $\epsilon = A/2$ such that the left hand of the above equation is always positive, which means that the integral is not convergent.

If A is zero, the solution is given by

$$u(x,t) = T_0 + \int_0^\infty B(\lambda)\sin(\lambda x)\exp(-\lambda k^2 t)d\lambda$$