MIDTERM II; MAT 312 (SPRING, 08)

Instructions: Do problems 1,2,3 below; also do one of problems 4 or 5 (not both).

(1) Let X denote a set and let $P(X)$ denote the collection of all subsets of X. For a given $S \in P(X)$ define a relation $\sim$ on $P(X)$ as follows: for any $A, B \in P(X)$ we have that $A \sim B$ iff $(A \cap B^c) \cup (B \cap A^c) \subset S$. Prove that $\sim$ is an equivalence relation on $P(X)$.
Solution: We must show that the relation is reflexive, symmetric and transitive.

Reflexive: For any $A \in P(X)$ note that $A \cap A^c = \emptyset$ and $\emptyset \subset S$. Thus $(A \cap A^c) \cup (A \cap A^c) \subset S$, showing that $A \sim A$.

Symmetry: For any $A, B \in P(X)$ we have that

$$(A \cap B^c) \cup (B \cap A^c) = (B \cap A^c) \cup (A \cap B^c).$$

Thus we have that $A \sim B \quad \Rightarrow \quad (A \cap B^c) \cup (B \cap A^c) \subset S \quad \Rightarrow \quad (B \cap A^c) \cup (A \cap B^c) \subset S \quad \Rightarrow \quad B \sim A$.

Transitive: For any $A, B, C \in P(X)$ note that

$$(A \cap B^c) \cup (B \cap A^c) \cup (B \cap C^c) \cup (C \cap B^c) = (A \cup B \cup C) \cap (A \cap B \cap C)^c$$

and

$$(A \cap C^c) \cup (C \cap A^c) = (A \cup C) \cap (A \cap C)^c \subset (A \cup B \cup C) \cap (A \cap B \cap C)^c.$$

Thus we have that $A \sim B$ and $B \sim C \Rightarrow (A \cup B \cup C) \cap (A \cap B \cap C)^c \subset S$ $\Rightarrow (A \cup C) \cap (A \cap C)^c \subset S \Rightarrow A \sim C$.

(2) Set $X = \{2, 4, 5, 6, 8, 10, 12, 24\}$ and define a relation R on X by
$xRy \quad \Leftrightarrow \quad 2x \mid y$

(a) Show that R is a strict partial ordering on X.
Solution: Must show that R is antisymmetric and transitive.

Antisymmetric: Suppose that xRy and yRx; then $y = 2xm$ and $x = 2yn$. Substituting $2xm$ for y in $x = 2yn$ we get that $x = 4xmn$, where x, m, n are possitive integers; dividing by x we get that 1=4mn, which is impossible. This contradiction shows that if xRy is true then yRx is not true.

Transitive: Suppose that xRy and yRz; thus $y = 2xm$ and $z = 2yn$. Substituting $2xm$ for y in $z = 2yn$ we get $z = (2x)(2mn)$. Thus $2x \mid z$ so xRz.

(b) Sketch the Hasse diagram for this relation.

The bottom line of the Hasse diagram consists of 2,5,6; the next line up consists of 4,10,12; and the top line constists of 8,24. There is a verticle line segment between the following pair of numbers: $(2,4),(5,10),(6,12),(2,12)$, $(4,8),(12,24),(4,24)$.

(3) Set $\sigma = (8,5,2)(3,5,7)(1,2,8,6,4)$; thus σ is in the permutation group on 8 letters $S(8)$.

(a) Compute σ^{-2}.
Solution: Since σ is represented by the matrix
1 2 3 4 5 6 7 8
8 5 2 1 7 4 3 6
it follows that σ^{-1} is represented by
1 2 3 4 5 6 7 8
4 3 7 6 2 8 5 1
and since $\sigma^{-2} = (\sigma^{-1})^2$ it follows that σ^{-2} is represented by
1 2 3 4 5 6 7 8
6 7 5 8 3 1 2 4 n

(b) Write σ as a product of disjoint cycles.
Solution: $\sigma = (1,8,6,4)(2,5,7,3)$

(c) Show that $\sigma^{90} = \sigma^{-2}$.
Solution: $order(\sigma) = lcm(4,4) = 4$; thus $\sigma^4 = id$. $\sigma^{90} = \sigma^{92}\sigma^{-2} = (\sigma^4)^{23}\sigma^{-2} = (id)^{23}\sigma^{-2} = \sigma^{-2}$.

(d) Compute $sgn(\sigma)$.
Solution: $sgn(\sigma) = sgn((1,8,6,4))sgn((2,5,7,3)) = (-1)^3(-1)^3 = +1$.

(4) For any positive integer n let $S^+(n)$ denote the permuations in $S(n)$ which have sign equal $+1$, and let $S^-(n)$ denote the permuations which have sign equal -1. Prove that the $S^+(n)$ and $S^-(n)$ have the same number of elements. (Hint: first show that the map $f : S(n) \longrightarrow S(n)$, defined by $f(\sigma) = (1,2)\sigma$ for all $\sigma \in S(n)$, maps $S^+(n)$ into $S^-(n)$.)

Solution: If $sgn(\sigma) = +1$ then $sgn(f(\sigma)) = sgn((1,2))sgn(\sigma) = (-1)(+1) = -1$; so f maps $S^+(n)$ into $S^-(n)$.

It will suffice to show that $f : S^+(n) \longrightarrow S^-(n)$ is a bijective map.

To see that f is one-one suppose that $f(\sigma) = f(\tau)$ for $\sigma, \tau \in S^+(n)$; this means that

$$(1,2)\sigma = (1,2)\tau.$$

If we multiply each side of the preceeding equation by $(1,2)$ (on the left) then we get

$$(1,2)^2\sigma = (1,2)^2\tau$$

which becomes

$$\sigma = \tau$$

because $(1,2)^2 = id$.

To see that $f : S^+(n) \longrightarrow S^-(n)$ is onto, choose any $\tau \in S^-(n)$ and set $\sigma = (1,2)\tau$. Note that $sgn(\sigma) = sgn((1,2))sgn(\tau) = (-1)(-1) = +1$; so $\sigma \in S^+(n)$. Note also that $f(\sigma) = f((1,2)\tau) = (1,2)^2\tau = \tau$ (because $(1,2)^2 = id$).

(5) Suppose that the permutations $\sigma, \tau \in S(n)$ are both transpositions which commute (i.e. $\sigma\tau = \tau\sigma$). Then show that either σ and τ are disjoint permutations, or $\sigma = \tau$.

Solution: $\sigma = (a, b)$ and $\tau = (c, d)$ for some integers $1 \leq a, b, c, d \leq n$ with $a \neq b$ and $c \neq d$.

Case I: Suppose that $a = c, b = d$ or $a = d, b = c$. In this case the permutations (a, b) and (c, d) are equal.

Case II: Suppose that $a \neq c, a \neq d, b \neq c, b \neq d$. In this case the permuations (a, b) and (c, d) are disjoint.

Case III: Suppose that If σ and τ are not disjoint and not equal; for example suppose that $a = c, b \neq c, b \neq d$. Then $\sigma\tau = (a, b)(a, d) = (a, d, b)$ and $\tau\sigma = (a, d)(a, b) = (a, b, d)$. Since $(a, d, b) \neq (a, b, d)$ it follows that $\sigma\tau \neq \tau\sigma$. This contradicts our assumption that σ and τ commute. This contradiction shows that case III does not occur.