**Instructions:** Do problems 1, 2, 3 below; also do one of problems 4 or 5 (not both).

(1) Let $X$ denote a set and let $P(X)$ denote the collection of all subsets of $X$. For a given $S \in P(X)$ define a relation $\sim$ on $P(X)$ as follows: for any $A, B \in P(X)$ we have that $A \sim B$ iff $(A \cap B^c) \cup (B \cap A^c) \subseteq S$. Prove that $\sim$ is an equivalence relation on $P(X)$.

**Solution:** We must show that the relation is reflexive, symmetric and transitive.

Reflexive: For any $A \in P(X)$ note that $A \cap A^c = \emptyset$ and $\emptyset \subseteq S$. Thus $(A \cap A^c) \cup (A \cap A^c) \subseteq S$, showing that $A \sim A$.

Symmetry: For any $A, B \in P(X)$ we have that $(A \cap B^c) \cup (B \cap A^c) = (B \cap A^c) \cup (A \cap B^c)$. Thus we have that $A \sim B \Rightarrow (A \cap B^c) \cup (B \cap A^c) \subseteq S \Rightarrow B \sim A$.

Transitive: For any $A, B, C \in P(X)$ note that

$$(A \cap B^c) \cup (B \cap A^c) \cup (B \cap C^c) \cup (C \cap B^c) = (A \cup B \cup C) \cap (A \cap B \cap C)^c$$

and

$$(A \cap C^c) \cup (C \cap A^c) = (A \cup C) \cap (A \cap C)^c \subseteq (A \cup B \cup C) \cap (A \cap B \cap C)^c.$$ 

Thus we have that $A \sim B$ and $B \sim C \Rightarrow (A \cup B \cup C) \cap (A \cap B \cap C)^c \subseteq S \Rightarrow (A \cup C) \cap (A \cap C)^c \subseteq S \Rightarrow A \sim C$.

(2) Set $X = \{2, 4, 5, 6, 8, 10, 12, 24\}$ and define a relation $R$ on $X$ by $xRy \iff 2x \mid y$.

(a) Show that $R$ is a strict partial ordering on $X$.

**Solution:** Must show that $R$ is antisymmetric and transitive.

Antisymmetric: Suppose that $xRy$ and $yRx$; then $y = 2xm$ and $x = 2yn$. Substituting $2xm$ for $y$ in $x = 2yn$ we get that $x = 4xmn$, where $x, m, n$ are positive integers; dividing by $x$ we get that $1 = 4mn$, which is impossible. This contradiction shows that if $xRy$ is true then $yRx$ is not true.

Transitive: Suppose that $xRy$ and $yRz$; thus $y = 2xm$ and $z = 2yn$. Substituting $2xm$ for $y$ in $z = 2yn$ we get $z = (2x)(2mn)$. Thus $2x \mid z$ so $xRz$.

(b) Sketch the Hasse diagram for this relation.

The bottom line of the Hasse diagram consists of 2, 5, 6; the next line up consists of 4, 10, 12; and the top line consists of 8, 24. There is a vertical line
segment between the following pair of numbers: \((2, 4), (5, 10), (6, 12), (2, 12), (4, 8), (12, 24), (4, 24)\).

(3) Set \(\sigma = (8, 5, 2)(3, 5, 7)(1, 2, 8, 6, 4)\); thus \(\sigma\) is in the permutation group on 8 letters \(S(8)\).

(a) Compute \(\sigma^{-2}\).

Solution: Since \(\sigma\) is represented by the matrix
\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
8 & 5 & 2 & 1 & 7 & 4 & 3 & 6
\end{array}
\]
it follows that \(\sigma^{-1}\) is represented by
\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
4 & 3 & 7 & 6 & 2 & 8 & 5 & 1
\end{array}
\]
and since \(\sigma^{-2} = (\sigma^{-1})^2\) it follows that \(\sigma^{-2}\) is represented by
\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
6 & 7 & 5 & 8 & 3 & 1 & 2 & 4
\end{array}
\]
(b) Write \(\sigma\) as a product of disjoint cycles.

Solution: \(\sigma = (1, 8, 6, 4)(2, 5, 7, 3)\)

(c) Show that \(\sigma^{90} = \sigma^{-2}\).

Solution: order(\(\sigma\)) = \(lcm(4, 4) = 4\); thus \(\sigma^4 = id\). \(\sigma^{90} = \sigma^{92}\sigma^{-2} = (\sigma^4)^{23}\sigma^{-2} = (id)^{23}\sigma^{-2} = \sigma^{-2}\).

(d) Compute \(sgn(\sigma)\).

Solution: \(sgn(\sigma) = sgn((1, 8, 6, 4))sgn((2, 5, 7, 3)) = (-1)^3(-1)^3 = +1\).

(4) For any positive integer \(n\) let \(S^+(n)\) denote the permutations in \(S(n)\) which have sign equal +1, and let \(S^-(n)\) denote the permutations which have sign equal -1. Prove that the \(S^+(n)\) and \(S^-(n)\) have the same number of elements. (Hint: first show that the map \(f : S(n) \rightarrow S(n)\), defined by \(f(\sigma) = (1, 2)\sigma\) for all \(\sigma \in S(n)\), maps \(S^+(n)\) into \(S^-(n)\).)

Solution: If \(sgn(\sigma) = +1\) then \(sgn(f(\sigma)) = sgn((1, 2))sgn(\sigma) = (-1)(+1) = -1\); so \(f\) maps \(S^+(n)\) into \(S^-(n)\).

It will suffice to show that \(f : S^+(n) \rightarrow S^-(n)\) is a bijective map.

To see that \(f\) is one-one suppose that \(f(\sigma) = f(\tau)\) for \(\sigma, \tau \in S^+(n)\); this means that
\[(1, 2)\sigma = (1, 2)\tau.
\]
If we multiply each side of the preceding equation by \((1, 2)\) (on the left) then we get
\[(1, 2)^2\sigma = (1, 2)^2\tau
\]
which becomes
\[\sigma = \tau
\]
because \((1, 2)^2 = id\).

To see that \(f : S^+(n) \rightarrow S^-(n)\) is onto, choose any \(\tau \in S^-(n)\) and set \(\sigma = (1, 2)\tau\). Note that \(sgn(\sigma) = sgn((1, 2))sgn(\tau) = (-1)(-1) = +1\); so \(\sigma \in S^+(n)\). Note also that \(f(\sigma) = f((1, 2)\tau) = (1, 2)^2\tau = \tau\) (because \((1, 2)^2 = id\).
(5) Suppose that the permutations $\sigma, \tau \in S(n)$ are both transpositions which commute (i.e. $\sigma\tau = \tau\sigma$). Then show that either $\sigma$ and $\tau$ are disjoint permutations, or $\sigma = \tau$.

Solution: $\sigma = (a, b)$ and $\tau = (c, d)$ for some integers $1 \leq a, b, c, d \leq n$ with $a \neq b$ and $c \neq d$.

Case I: Suppose that $a = c, b = d$ or $a = d, b = c$. In this case the permutations $(a, b)$ and $(c, d)$ are equal.

Case II: Suppose that $a \neq c, a \neq d, b \neq c, b \neq d$. In this case the permutations $(a, b)$ and $(c, d)$ are disjoint.

Case III: Suppose that If $\sigma$ and $\tau$ are not disjoint and not equal; for example suppose that $a = c, b \neq c, b \neq d$. Then $\sigma\tau = (a, b)(a, d) = (a, d, b)$ and $\tau\sigma = (a, d)(a, b) = (a, b, d)$. Since $(a, d, b) \neq (a, b, d)$ it follows that $\sigma\tau \neq \tau\sigma$. This contradicts our assumption that $\sigma$ and $\tau$ commute. This contradiction shows that case III does not occur.