Homework 11 : §5.3 - $1^*, 3, 4^*, 5, 7^*, 8^*$.
$*$ - in the back of the book

Exercises 5.3

3. Let G be a group with the map $\phi : G \to G, g \mapsto aga^{-1}$. It is a homomorphism since

$$\phi(g_1 g_2) = ag_1 g_2 a^{-1} = ag_1 a^{-1} ag_2 a^{-1} = \phi(g_1)\phi(g_2).$$

It is injective since $ag_1 a^{-1} = ag_2 a^{-1}$ implies $g_1 = g_2$. It is surjective since given $g \in G$, $\phi(a^{-1}ga) = g$. Thus, ϕ is an isomorphism.

5. $G \times H$ is abelian if and only if $(g_1, h_1) \cdot (g_2, h_2) = (g_2, h_2) \cdot (g_1, h_1)$ for any $g_1, g_2 \in G$ and $h_1, h_2 \in H$. But $(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$ and $(g_2, h_2) \cdot (g_1, h_1) = (g_2 g_1, h_2 h_1)$. Thus, $(g_1 g_2, h_1 h_2) = (g_2 g_1, h_2 h_1)$ if and only if $g_1 g_2 = g_2 g_1, h_1 h_2 = h_2 h_1$, which is tantamount to G, H being abelian groups.