Homework 5 : §2.1 - $3, 4, 7, 8, 9$, §2.2 - $2, 6, 7, 8, 10, 11$.

Exercises 2.1

3. We have
$$Y^c = U \setminus Y = (X \cup Y) \setminus Y = X \setminus Y.$$
Since $X \setminus Y \subset X$, we get
$$X \cap Y^c = X \cap (X \setminus Y) = X \setminus Y.$$

4. The picture of $A\Delta B$ can be represented by a Venn diagram.
$$\begin{aligned}
(A\Delta B)\Delta C &= ((A \setminus B) \cup (B \setminus A))\Delta C \\
&= ((A \cap B^c) \cup (A^c \cap B))\Delta C \\
&= ((A \cap B^c) \cup (A^c \cap B)) \cap C^c \cup ((A \cap B^c) \cup (A^c \cap B))^c \cap C \\
&= (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup ((A \cap B^c)^c \cap (A^c \cap B)^c \cap C) \\
&= (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup ((A^c \cup B) \cap (A \cup B^c) \cap C) \\
&= (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C) \cup (A \cap B \cap C).
\end{aligned}$$
Since the answer is symmetric with respect to A, B, C, this means the ordering in $(A\Delta B)\Delta C$ is unimportant, i.e.,
$$(A\Delta B)\Delta C = (B\Delta C)\Delta A = A\Delta(B\Delta C).$$

7. (i) We have the following equivalences :
$$\begin{aligned}
(x, y) \in (A \times C) \cap (B \times D) &\Leftrightarrow (x, y) \in A \times C, (x, y) \in B \times D \\
&\Leftrightarrow x \in A \cap B, y \in C \cap D \\
&\Leftrightarrow (x, y) \in (A \cap B) \times (C \cap D).
\end{aligned}$$
(ii) This is FALSE. For a counterexample set $A = [0, 1] = C, B = [0, 1/2], D = [0, 2]$.

8. The set $X \times Y$ has mn elements.

9. There are plenty of examples. Any conic (i.e., parabola, ellipse, circle, a pair of straight lines) is an example. More specifically, set $A = \mathbb{R} = B, X = \{(x, y)|x^2 + y^2 = 1\}$.

Exercises 2.2

2. (i) bijective (ii) surjective but NOT injective (iii) neither (iv) surjective but NOT injective (v) injective but NOT surjective.

6. Since a bijection from $X = \{0, 1, 2\}$ to itself is a map which maps injectively, 0 has three choices for its image. Once that is chosen, 1 has two choices for its image. After choosing, 2 has only one choice (not a choice after all!) as its image. Thus, the total possible bijections are $3 \cdot 2 \cdot 1 = 6$. More generally, the number of bijections between two sets of size n is $n!$.

7. (i) Since $f(x) = (4-x)/3$, we get $x = (4-f^{-1}(x))/3$. Rearranging we have $f^{-1}(x) = 4-3x$.

(ii) Notice that $f(x) = x^3 - 3x^2 + 3x - 1 = (x - 1)^3$. Thus, $x = (f^{-1}(x) - 1)^3$, whence $f^1(x) = x^{1/3} + 1$.

8. Drawing a Venn diagram is perhaps the quickest proof (and also reveals the identity). To prove it algebraically we repeatedly use

$$|X \cup Y| = |X| + |Y| - |X \cap Y|.$$

Thus,

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B \cup C| - |A \cap (B \cup C)| \\ &= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)| \\ &= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|. \end{aligned}$$

10. (a) Follows from the definition of $\chi_A(x)$.

(b) Let $f : X \to \{0, 1\}$ be given. Define $A = \{x \in X | f(x) = 1\}$. Then it is easily verified that $f \equiv \chi_A$.

11. We claim that for a set X with $|X| = n$ the power set $\mathcal{P}(x)$ has 2^n elements. When $X = \{1\}, \mathcal{P}(X) = \{\phi, \{1\}\}$ and it has 2 elements. Suppose this is true for sets of size up to n. Let X' be a set of size $n + 1$. We can write X' as the disjoint union of a set X of size n and an element α. Then any element Γ of $\mathcal{P}(X')$ either contains α, in which case Γ is a subset of X union α, or it doesn't contain α, in which case Γ is a subset of X. Thus, every element of $\mathcal{P}(x)$ 'appears' twice in $\mathcal{P}(X')$, whence $|\mathcal{P}(X')| = 2|\mathcal{P}(X)| = 2^{n+1}$. Then by the principle of induction we are done.