\textbf{Homework 5} : §2.1 - 3, 4, 7, 8, 9, §2.2 - 2, 6, 7, 8, 10, 11.

\textbf{Exercises 2.1}

3. We have
\[ Y^c = U \setminus Y = (X \cup Y) \setminus Y = X \setminus Y. \]
Since \( X \setminus Y \subset X \), we get
\[ X \cap Y^c = X \cap (X \setminus Y) = X \setminus Y. \]

4. The picture of \( A \Delta B \) can be represented by a Venn diagram.
\[
(A \Delta B) \Delta C = ((A \setminus B) \cup (B \setminus A)) \Delta C
\]
\[ = ((A \cap B^c) \cup (A^c \cap B)) \Delta C
\]
\[ = ((A \cap B^c) \cup (A^c \cap B)) \cap C^c \cup ((A \cap B^c) \cup (A^c \cap B))^c \cap C
\]
\[ = (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup ((A \cap B^c) \cap (A^c \cap B)^c \cap C)
\]
\[ = (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A \cap B \cap C).
\]
Since the answer is symmetric with respect to \( A, B, C \), this means the ordering in \( (A \Delta B) \Delta C \)

\( (A \Delta B) \Delta C = (B \Delta C) \Delta A = A \Delta (B \Delta C). \)

7. (i) We have the following equivalences :
\[ (x, y) \in (A \times C) \cap (B \times D) \iff (x, y) \in A \times C, (x, y) \in B \times D
\]
\[ \iff x \in A \cap B, y \in C \cap D
\]
\[ \iff (x, y) \in (A \cap B) \times (C \cap D).
\]
(ii) This is FALSE. For a counterexample set \( A = [0, 1] = C, B = [0, 1/2], D = [0, 2]. \)

8. The set \( X \times Y \) has \( mn \) elements.

9. There are plenty of examples. Any conic (i.e., parabola, ellipse, circle, a pair of straight lines) is an example. More specifically, set \( A = \mathbb{R} = B, X = \{(x, y)|x^2 + y^2 = 1 \}. \)

\textbf{Exercises 2.2}

2. (i) bijective (ii) surjective but NOT injective (iii) neither (iv) surjective but NOT injective
(v) injective but NOT surjective.

6. Since a bijection from \( X = \{0, 1, 2\} \) to itself is a map which maps injectively, 0 has three choices for its image. Once that is chosen, 1 has two choices for its image. After choosing, 2 has only one choice (not a choice after all!) as its image. Thus, the total possible bijections are \( 3 \cdot 2 \cdot 1 = 6 \). More generally, the number of bijections between two sets of size \( n \) is \( n! \).

7. (i) Since \( f(x) = (4-x)/3 \), we get \( x = (4- f^{-1}(x))/3 \). Rearranging we have \( f^{-1}(x) = 4-3x \).
(ii) Notice that \( f(x) = x^3 - 3x^2 + 3x - 1 = (x - 1)^3 \). Thus, \( x = (f^{-1}(x) - 1)^3 \), whence \( f^1(x) = x^{1/3} + 1. \)
8. Drawing a Venn diagram is perhaps the quickest proof (and also reveals the identity). To prove it algebraically we repeatedly use

$$|X \cup Y| = |X| + |Y| - |X \cap Y|.$$ 

Thus,

$$|A \cup B \cup C| = |A| + |B \cup C| - |A \cap (B \cup C)|$$
$$= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|$$
$$= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|.$$ 

10. (a) Follows from the definition of $\chi_A(x)$.
(b) Let $f : X \to \{0, 1\}$ be given. Define $A = \{x \in X | f(x) = 1\}$. Then it is easily verified that $f \equiv \chi_A$.

11. We claim that for a set $X$ with $|X| = n$ the power set $\mathcal{P}(X)$ has $2^n$ elements. When $X = \{1\}$, $\mathcal{P}(X) = \{\emptyset, \{1\}\}$ and it has 2 elements. Suppose this is true for sets of size up to $n$. Let $X'$ be a set of size $n + 1$. We can write $X'$ as the disjoint union of a set $X$ of size $n$ and an element $\alpha$. Then any element $\Gamma$ of $\mathcal{P}(X')$ either contains $\alpha$, in which case $\Gamma$ is a subset of $X$ union $\alpha$, or it doesn’t contain $\alpha$, in which case $\Gamma$ is a subset of $X$. Thus, every element of $\mathcal{P}(x)$ ‘appears’ twice in $\mathcal{P}(X')$, whence $|\mathcal{P}(X')| = 2|\mathcal{P}(X)| = 2^{n+1}$. Then by the principle of induction we are done.