MAT 310-F10: REVIEW FOR FINAL EXAM

(1) Consider the the 3×6 matrix over the real numbers $A = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6]$, where $\mathbf{a}_i$ denotes the i'th column. Let B denote the 3×6 matrix (over the real numbers)

$$\begin{matrix} 0 & 1 & 2 & 0 & 7 & 6 \\ 1 & 0 & 3 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 1 \end{matrix}$$

(a) Suppose $\mathbf{a}_2 = (1,2,2)^t$, $\mathbf{a}_3 = (-2,0,1)^t$, $\mathbf{a}_4 = (0,4,5)^t$, $\mathbf{a}_5 = (0,1,1)^t$. Compute the ranks of A and B. Explain why B can not be obtained from A by a finite number of elementary row operations.
Solution: The first 3 columns of B are independent, so its column space has dimension 3, thus rank(B)=3. The second, third and fifth column of A are independent, so its column space has dimension 3, thus rank(A)=3.
If B could be obtained from A by elementary row operations, then there would exist an invertible, 3×3-matrix C such that $C\mathbf{b}_i = \mathbf{a}_i$ holds for all $1 \le i \le 6$ ($\mathbf{b_i}$ denotes the i'th column of B). Since the $\{\mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ are idependent, and left multiplication by an invertible matrix C sends an independent set to an independent set, it would follow that $\{\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ must be independent — which it is not.

(b) Suppose that $\mathbf{a}_2 = (1,1,1)^t$, $\mathbf{a}_4 = (1,0,5)^t$, $\mathbf{a}_6 = (1,2,3)^t$; also suppose that B is obtained from A by a finite number of elementary row operations. Then compute the coordinates of $\mathbf{a}_3$.
Solutions: We have that $\mathbf{a}_i = C\mathbf{b}_i$ holds for all $1 \le i \le 6$ for some invertible matrix C. Note that $\mathbf{b}_3 = -10\mathbf{b}_2 + \mathbf{b}_4 + 2\mathbf{b}_6$. Thus $\mathbf{a}_3 = C\mathbf{b}_3 = -10C\mathbf{b}_2 + C\mathbf{b}_4 + 2C\mathbf{b}_6 = -10\mathbf{a}_2 + \mathbf{a}_4 + 2\mathbf{a}_6 = (-7,-6,1)^t$.

Hint: read the proof of Theorem 3.16 on page 190.

(2) Consider the following 3×3 matrix A (over the real numbers)

$$\begin{matrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{matrix}$$

(a) Compute the determinant for A, det(A)=?
Solution: det(A)=-9

(b) Compute the characteristic polynomial of A, $p_A(t) =$?
Solution: $p_A(t) = -t^3 + 5t^2 - 3t - 9$

(c) Compute eigenvalues for A; for each eigenvalue λ compute its mulitplicity and find a basis for the eigenspace E_λ.
Solution: $p_A(t) = -(t+1)(t-3)^2$ so the eigenvalues are -1,3 having multiplicity 1,2 respectively. A basis for E_{-1} is $\{(2,4,3)^t\}$. A basis for E_3 is $\{(1,1,0)^t, (0,0,1)^t\}$.

(d) Diagonalize A; that is write $Q^{-1}AQ = D$, where D is a diagonal matrix.
Solution: D is the matrix

$$\begin{matrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{matrix}$$

Q is the matrix

$$\begin{matrix} 2 & 1 & 0 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{matrix}$$

(e) Compute A^{99}=? (**Hint:** If $A = QDQ^{-1}$ then $A^n = QD^nQ^{-1}$ for any positive integer n.)
Solution: Note that D^n is the matrix

$$\begin{matrix} (-1)^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 3^n \end{matrix}$$

So A^n is the product of the 3 matrices QD^nQ^{-1}.

(3) Define a linear transformation $T : P_3(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$ by $T(f(x)) = xf'(x) + f''(x) - f(2)$ for each polynomial $f(x) \in P_3(\mathbb{R})$. ;

(a) Compute det(T) and the characteristic polynomial $p_T(t)$ for T.
Solution: If α denotes the standard basis $\{1, x, x^2, x^3\}$ for $P_3(\mathbb{R})$ then $[T]_\alpha$ is the matrix

$$\begin{matrix} -1 & -2 & -2 & -8 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{matrix}$$

Since this is an upper triangular matrix the determinant is the product of the diagonal elements

$$det([T]_\alpha) = (-1)(1)(2)(3) = -6.$$

Likewise $p_{[T]_\alpha}(t) = det([T]_\alpha - tI_4) = (-1-t)(1-t)(2-t)(3-t)$. Finally note that $det(T) = det([T]_\alpha)$ and $p_T(t) = p_{[T]_\alpha}(t)$.

(b) Find all the eigenvalues for T; for each eigenvalue λ compute its multiplicity and find a basis for its eignspace E_λ.
Solution: The eigenvalues are -1,1,2,3. Each eigenvalue has multiplicity one.
A basis for E_{-1} is $\{1\}$; a basis for E_1 is $\{x-1\}$; a basis for E_2 is $\{x^2 - \frac{2}{3}\}$; a basis for E_3 is $\{x^3 + 3x - \frac{14}{4}\}$

(c) Find a basis for $P_3(\mathbb{R})$ consisting of eigenvectors for T.
Solution: The four vectors given in part (b) are such a basis.

(d) Compute $T^{45}(x^3) =$? (**Hint:** express the polynomial x^3 as a linear combination of the basis elements given in part (c) above.)
Solution: Note that

$$x^3 = (x^3 + 3x - \frac{14}{4}) - 3(x-1) + \frac{1}{2}(1)$$

. Thus

$$T^{45}(x^3) = T^{45}(x^3 + 3x - \frac{14}{4}) - 3T^{45}(x-1) + \frac{1}{2}T^{45}(1) =$$

$$3^{45}(x^3 + 3x - \frac{14}{4}) - 3(x-1) - \frac{1}{2}.$$

(4) A polynomial $f(x) \in P(F)$ is called *irreducible* over the field F if whenever $f(x) = g(x)h(x)$ for $g(x), h(x) \in P(F)$ then either $g(x) = \alpha$ or $h(x) = \alpha$ for some $\alpha \in F$.

Let V denote a finite dimensional vector space over the field F and let $T : V \longrightarrow V$ denote a linear transformation. Show that if the characteristic polynomial $P_T(t)$ for T is irreducible then V is a T-cyclic subspace (of itself) generated by some $\mathbf{v} \in V$. (**Hint:** *T-cyclic subspaces* are defined on page 313 in section 5.4; see also Theorem 5.21 on page 314.)
Solution: Choose any non-zero vector $\mathbf{v} \in V$, and let W denote the T-cyclic subspace of V generated by $\mathbf{v}$. Then W is also a T-invariant subspace of V (see section 5.4 of text), so Theorem 5.21 states that the characteristic polynomial $p_{T_W}(t)$ for T_W is a factor of the characteristic polynomial $p_T(t)$ for T. Since $p_T(t)$ is irreduciable we conclude that $p_{T_W}(t) = \alpha p_T(t)$ for some scalar α; hence $deg(p_{T_W}(t)) = deg(p_T(t))$, which implies that dim(W)=dim(V), which implies that W=V.

(5) Let F denote a field. Given $A \in M_{3\times3}(F)$, define a linear operator $T : M_{3\times3}(F) \longrightarrow M_{3\times3}(F)$ by $T(B) = AB$ for any $B \in M_{3\times3}(F)$. Explain why any T-cyclic subspace $W \subset M_{3\times3}(F)$ satisfies $dim(W) \leq 3$. (**Hint:** Cayley-Hamilton Theorem for matrices.)
Solution: Any T-cyclic subspace W has the form $span(B, AB, A^2B, A^3B, ..., A^nB, ...)$ for some $B \in M_{3\times3}(F)$. It will suffice to show that

$$(i) \qquad W = span(B, AB, A^2B).$$

Let $-t^3 + at^2 + bt + c$ denote the characteristic polynmial for the matrix A; then, by the matrix form of the Cayley-Hamilton theorem, we have

$$(ii) \qquad -A^3 + aA^2 + bA + cI_3 = 0.$$

Deduce from (ii) that

$$(iii) \qquad A^nB = aA^{n-1}B + bA^{n-2}B + cA^{n-3}B$$

for all $n \geq 3$. It follows from (iii) that

$$(iv) \qquad span(B, AB, A^2B, ..., A^{n-1}B) = span(B, AB, A^2, ..., A^nB)$$

for all $n \geq 3$. Thus by induction over n in (iv) we get that

$$(v) \qquad span(B, AB, A^2B) = span(B, AB, A^2, ..., A^nB)$$

for all $n \geq 3$.

(6) Let $T : V \longrightarrow V$ denote a linear operator on the finite dimensional vector space V over the field F; and let $id_V : V \longrightarrow V$ denote the identity map. For some $\mathbf{v} \in V$, $\lambda \in F$ and m a positive integer suppose that $(T - \lambda id_V)^{m-1}(\mathbf{v}) \neq \mathbf{0}$ but $(T - \lambda id_V)^m(\mathbf{v}) = \mathbf{0}$.

(a) Show that λ is an eigenvalue for T.
Solution: Set $\mathbf{w} = (T - \lambda id_V)^{m-1}(\mathbf{v})$; then $\mathbf{w} \neq \mathbf{0}$ and $T(\mathbf{w}) = \lambda\mathbf{w}$. Thus $\mathbf{w}$ is an eigenvector for T associated to the eigenvalue λ.

(b) Show that $\beta = \{(T - \lambda id_V)^i(\mathbf{v}) \mid i = 0, 1, 2, ..., m-1\}$ is an independent subset of V.
Solution: Suppose that

$$(i) \qquad \sum_{i=j}^{m-1} a_i(T - \lambda id_V)^i(\mathbf{v}) = \mathbf{0}$$

is a given linear relation with $a_j \neq 0$. By applying $(T - \lambda id_V)^{m-1-j}$ to both sides of (i) we get

$$(ii) \qquad a_j(T - \lambda id_V)^{m-1}(\mathbf{v}) = \mathbf{0}.$$

Since $(T - \lambda id_V)^{m-1}(\mathbf{v}) \neq \mathbf{0}$ it follows from (ii) that $a_j = 0$; this is a contradiction.

(c) Set $W = span(\beta)$. Explain why the subspace W is T-invariant.
Solution: Set $\mathbf{w}_i = (T - \lambda id_V)^{i-1}(\mathbf{v})$. Note that

$$(i) \qquad T(\mathbf{w}_i) = \mathbf{w}_{i+1} + \lambda\mathbf{w}_i.$$

So T maps the spanning set for W into W; thus $T(W) \subset W$.

(d) Explain why $(t - \lambda)^m$ is a factor of the characteristic polynomial of T; i.e. $p_T(t) = (t - \lambda)^m g(t)$ for some $g(t) \in P(F)$. (**Hint:** What is the characteristic polynomial $p_{T_W}(t)$ and why is it a factor of $p_T(t)$?)
Solution: W is a T invariant subspace of V (part (c)); so $p_{T_V}(t)$ divides $p_T(t)$ (theorem 5.21). β is a basis for W (part (b)); and the matrix $[T]_\beta$ is an $m \times m$ Jordan Block matrix having λ down the diagonal (see (i) in part (c)). Thus $p_{T_V}(t) = (-1)^m(t - \lambda)^m$.

(7) Let $T : V \longrightarrow V$ denote a linear operator on the real vector space V. Suppose that V is the direct sum $U \oplus W$ of T-invariant subspaces $U, W \subset V$. If λ is an eigenvalue for T, then show that either $\dim(E_\lambda \cap U) \geq 1$ or $\dim(E_\lambda \cap W) \geq 1$.

Solution: Every vector $\mathbf{v} \in V$ can be written uniquely as a sum of a vector in U with a vector in W; thus

$$(i) \qquad \mathbf{v} = \mathbf{u} + \mathbf{w}$$

where $\mathbf{u} \in U$ and $\mathbf{w} \in W$. From (i) we deduce

$$(ii) \qquad \lambda\mathbf{v} = \lambda\mathbf{u} + \lambda\mathbf{w}$$

and

$$(iii) \qquad T(\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{w}).$$

If $\mathbf{v} \in E_\lambda$ we have that

$$(iv) \qquad T(\mathbf{v}) = \lambda\mathbf{v}.$$

Note also that

$$(v) \qquad \lambda\mathbf{u}, T(\mathbf{u}) \in U \quad and \quad \lambda\mathbf{w}, T(\mathbf{w}) \in W.$$

Now, by (iv) and (v), equations (ii) and (iii) give two ways to write $\lambda\mathbf{v}$ as a sum of a vector in U with a vector in W; by uniqueness of such a summation it follows that

$$(vi) \qquad T(\mathbf{u}) = \lambda\mathbf{u} \quad and \quad T(\mathbf{w}) = \lambda\mathbf{w}.$$

Thus, assuming $\mathbf{v} \neq \mathbf{0}$, it follows from (vi) that either $\mathbf{u}$ or $\mathbf{w}$ is an eigenvector for T associated to λ.

(8) There will be a problem on the exam similiar to problem (2) or problem (3) at the end of section 7.1.