MAT 310-F10: REVIEW FOR FINAL EXAM

(1) Consider the the 3×6 matrix over the real numbers $A = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6]$, where $\mathbf{a}_i$ denotes the i'th column. Let B denote the 3×6 matrix (over the real numbers)

$$\begin{matrix} 0 & 1 & 2 & 0 & 7 & 6 \\ 1 & 0 & 3 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 1 \end{matrix}$$

(a) Suppose $\mathbf{a}_2 = (1, 2, 2)^t$, $\mathbf{a}_3 = (-2, 0, 1)^t$, $\mathbf{a}_4 = (0, 4, 5)^t$, $\mathbf{a}_5 = (0, 1, 1)^t$. Compute the ranks of A and B. Explain why B can not be obtained from A by a finite number of elementary row operations.

(b) Suppose that $\mathbf{a}_2 = (1, 1, 1)^t$, $\mathbf{a}_4 = (1, 0, 5)^t$, $\mathbf{a}_6 = (1, 2, 3)^t$; also suppose that B is obtained from A by a finite number of elementary row operations. Then compute the coordinates of $\mathbf{a}_3$.

Hint: read the proof of Theorem 3.16 on page 190.

(2) Consider the following 3×3 matrix A (over the real numbers)

$$\begin{matrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{matrix}$$

(a) Compute the determinant for A, det(A)=?

(b) Compute the characteristic polynomial of A, $p_A(t) =$?

(c) Compute eigenvalues for A; for each eigenvalue λ compute its mulitplicity and find a basis for the eigenspace E_λ.

(d) Diagonalize A; that is write $Q^{-1}AQ = D$, where D is a diagonal matrix.

(e) Compute A^{99}=? (**Hint:** If $A = QDQ^{-1}$ then $A^n = QD^nQ^{-1}$ for any positive integer n.)

(3) Define a linear transformation $T : P_3(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$ by $T(f(x)) = xf'(x) + f''(x) - f(2)$ for each polynomial $f(x) \in P_3(\mathbb{R})$. ;

(a) Compute det(T) and the characteristic polynomial $P_T(t)$ for T.

(b) Find all the eigenvalues for T; for each eigenvalue λ compute its multiplicity and find a basis for its eignspace E_λ.

(c) Find a basis for $P_3(\mathbb{R})$ consisting of eigenvectors for T.

(d) Compute $T^{45}(x^3)$ =? (**Hint:** express the polynomial x^3 as a linear combination of the basis elements given in part (c) above.)

(4) A polynomial $f(x) \in P(F)$ is called *irreducible* over the field F if whenever $f(x) = g(x)h(x)$ for $g(x), h(x) \in P(F)$ then either $g(x) = \alpha$ or $h(x) = \alpha$ for some $\alpha \in F$.

Let V denote finite dimensional vector space over the field F and let $T : V \longrightarrow V$ denote a linear transformation. Show that if the characteristic polynomial $P_T(t)$ for T is irreducible then V is a T-cyclic subspace (of itself) generated by some $\mathbf{v} \in V$. (**Hint:** *T-cyclic subspaces* are defined on page 313 in section 5.4; see also Theorem 5.21 on page 314.)

(5) Let F denote a field. Given $A \in M_{3\times 3}(F)$, define a linear operator $T : M_{3\times 3}(F) \longrightarrow M_{3\times 3}(F)$ by $T(B) = AB$ for any $B \in M_{3\times 3}(F)$. Explain why any T-cyclic subspace $W \subset M_{3\times 3}(F)$ satisfies $dim(W) \leq 3$. (**Hint:** Cayley-Hamilton Theorem for matrices.)

(6) Let $T : V \longrightarrow V$ denote a linear operator on the finite dimensional vector space V over the field F; and let $id_V : V \longrightarrow V$ denote the identity map. For some $\mathbf{v} \in V$, $\lambda \in F$ and m a positive integer suppose that $(T - \lambda id_V)^{m-1}(\mathbf{v}) \neq \mathbf{0}$ but $(T - \lambda id_V)^m(\mathbf{v}) = \mathbf{0}$.

(a) Show that λ is an eigenvalue for T.

(b) Show that $\beta = \{(T - \lambda id_V)^i(\mathbf{v}) \mid i = 0, 1, 2, ..., m-1\}$ is an independent subset of V.

(c) Set $W = span(\beta)$. Explain why the subspace W is T-invariant.

(d) Explain why $(t - \lambda)^m$ is a factor of the characteristic polynomial of T; i.e. $p_T(t) = (t - \lambda)^m g(t)$ for some $g(t) \in P(F)$. (**Hint:** What is the characteristic polynomial $p_{T_W}(t)$ and why is it a factor of $p_T(t)$?)

(7) Let $T : V \longrightarrow V$ denote a linear operator on the real vector space V. Suppose that V is the direct sum $U \oplus W$ of T-invariant subspaces $U, W \subset V$. If λ is an eigenvalue for T, then show that either $\dim(E_\lambda \cap U) \geq 1$ or $\dim(E_\lambda \cap W) \geq 1$.

(8) There will be a problem on the exam similiar to problem (2) or problem (3) at the end of section 7.1.