REVIEW FOR MAT 203 FINAL EXAM

(1) Set $f(x,y) = yln(y) + xy^2$, $\mathbf{u} = 2^{-1/2}\mathbf{i} + 2^{-1/2}\mathbf{j}$ and $\mathbf{v} = 2^{-1}3^{1/2}\mathbf{i} + 2^{-1}\mathbf{j}$.

(a) Compute the directional derivatives $D_{\mathbf{u}}f(1,2)$ and $D_{\mathbf{v}}f(1,2)$.
Solution: using 13.10 on page 934 we get that

$$D_{\mathbf{u}}f(1,2) =< 4, 5 + ln(2) > \bullet < 2^{-1/2}, 2^{-1/2} >= (ln(2)+9)/2^{1/2}$$

$$D_{\mathbf{v}}f(1,2) =< 4, 5 + ln(2) > \bullet < 3^{1/2}/2, 1/2 >= (4(3^{1/2}) + 5 + ln(2))/2$$

(b) For which unit vector $\mathbf{w}$ is the directional derivative $D_{\mathbf{w}}f(1,2)$ maximal?
Solution: $\mathbf{w} = grad(f)(1,2)/ \parallel grad(f)(1,2) \parallel$ (see 13.11 on page 935).

(2) Find an equation for the tangent plane to the surface given by $z^2 + 3x^2 - y^2 = 3$ at the point $(0,1,2)$.
Solution: Set $f(x,y,z) = z^2 + 3x^2 - y^2$; the vector $grad(f)(0,1,2) = -2\mathbf{j} + 4\mathbf{k}$ is the normal direction to the tangent plane. Thus an equation for the tangent plane is

$$-2(y-1) + 4(x-2) = 0$$

(3) Set $f(x,y) = -y^3 + x^2 - xy - 1$.

(a) Find all the critical points of $f(x,y)$.
Solution: $grad(f)(x,y) =< 2x - y, -3y^2 - x >$; the critical points are the solutions to $grad(f)(x,y) =< 0, 0 >$; thus the critical points are $(0,0), (-1/12, -1/6)$.

(b) Use the second derivative test to determine the nature of these critical points (local max., local min., saddle point).
Solution: $f_{xx}(0,0)f_{yy}(0,0) - (f_{xy}(0,0))^2 = 0 - 1 < 0$; so $(0,0,f(0,0))$ is a saddle point. $f_{xx}(-1/12,-1/6) = 2 > 0$ and
$f_{xx}(-1/12,-1/6)f_{yy}(-1/12,-1/6) - (f_{xy}(-1/12,-1/6))^2 = 2 - 1 = 1 > 0$; so $f(-1/12,-1/6)$ is a relative minimum value.

(c) Find the maximum and minimum values for $f(x,y)$ on the region described by the following inequalities:

$$-1 \leq x \leq 1$$

$$-1 \leq y \leq 1 \quad .$$

Hint: The region is a square. To find the maximum value M that f takes on this square let m_1, m_2, m_3, m_4 denote the maximum values that f takes on each of the four edges of the square; then M is the maximum of all the numbers $m_1, m_2, m_3, m_4, f(0,0), f(-1/12,-1/6)$.

(d) Does $f(x,y)$ take on an absolute maximum value or an absolute minimum value on the whole plane?
Solution: Note that $f(0,y) = -y^3 - 1$ which has neither a maximum nor a minimum value. So f does not take on an absolute maximum value or an absolute minimum value on the plane.

(4) problem#19 page 965
Solution: Find the critical points of the profit function $P(x_1, x_2)$; that is, find the solutions to the equation

$$grad(P)(x_1, x_2) = < 0, 0 > \quad .$$

The first and second coordinates of this equation are

$$11 - .04x_1 = 0$$

$$11 - .1x_2 = 0$$

respectively; these two coordinate equations have solutions $x_1 = 275$ and $x_2 = 110$ respectively.

(5) problem #6 page 974
Solution: Set $f(x,y) = x^2 - y^2$ and $g(x,y) = 2y - x^2$. We must first solve the equations $grad(f)(x,y) = \lambda grad(g)(x,y)$ and $g(x,y) = 0$ for x, y. The first and second coordinates of the first of these equations are

$$2x = -2\lambda x$$

$$-2y = 2\lambda$$

respectively, and from $g(x,y) = 0$ we get that

$$y = x^2/2 \quad .$$

One set of solutions to these equations is

$$x = 0, y = 0, \lambda = anything \quad ;$$

if $x \neq 0$ then from the first of the above three equations we deduce that

$$\lambda = -1$$

which allows to deduce from the second and third of these equations that

$$y = 1$$

$$x = 2^{1/2}, -2^{1/2} \quad .$$

Finally the desired maximum value for $f(x,y)$ — subject to the constraint $g(x,y) = 0$ — is the maximum of the 3 values

$$f(0,0) = 0, f(2^{1/2}, 1) = 1, f(-2^{1/2}, 1) = 1 \quad .$$

(6) problem #35 page 975

(7) Evaluate the following integrals.

(a) $\int_0^1 \int_0^x (1-x^2)^{1/2} dydx$.
Solution: $\int_0^x (1-x)^2)^{1/2} dy = y(1-x^2)^{1/2} \|_0^x = x(1-x^2)^{1/2}$; and $\int_0^1 x(1-x^2)^{1/2} dx = (-1/3)(1-x^2)^{3/2} \|_0^1 = 0 - (-1/3) = 1/3$.

(b) $\int\int_R e^{-x^2-y^2} dA$, where R is the region in the plane described by

$$0 \leq x^2 + y^2 \leq 25$$

$$0 \leq x, y \quad .$$

Solution: Using polar coordintes we have that

$$\int\int_R e^{-x^2-y^2} dA = \int_0^{2\pi} \int_0^5 e^{-r^2} r dr d\theta \quad .$$

Note that

$$\int_0^5 e^{-r^2} r dr = (-1/2) e^{-r^2} \|_0^5 = (-1/2e^{25}) + 1/2$$

and

$$\int_0^{2\pi} (-1/2e^{25}) + 1/2 d\theta = (-\pi/e^{25}) + \pi \quad .$$

(8) Let Q denote the region in 3-space desribed by

$$0 \leq y \leq 9$$

$$0 \leq x \leq y/3$$

$$0 \leq z \leq (y^2 - 9x^2)^{1/2} \quad .$$

Let $\rho(x,y,z) = z$ denote a given density function for the region Q.

(a) Sketch the region Q.

(b) Find the mass of Q.
Solution: $mass = \int_0^9 \int_0^{y/3} \int_0^{(y^2-9x^2)^{1/2}} \rho(x,y,z) dzdxdy$, where $\rho(x,y,z) = z$. Note that

$$\int_0^{(y^2-9x^2)^{1/2}} zdz = z^2/2 \|_0^{(y^2-9x^2)^{1/2}} = (y^2 - 9x^2)/2$$

and

$$\int_0^{y/3} (y^2 - 9x^2)/2 dx = 1/2(y^2 x - 3x^3 \|_0^{y/3}) = y^3/9$$

and

$$\int_0^9 y^3/9 dy = y^4/36 \|_0^9 = 9^4/36 \quad .$$

(c) Find the y-coordinate of the center of mass for Q.

Solution: The y-coordinate of the center of mass is equal to the quotient $M_{x,z}/mass$, where $M_{x,y}$ is the "first moment" of the region Q about the x, z plane defined to be the triple integral

$$\int_0^9 \int_0^{y/3} \int_0^{(y^2-9x^2)^{1/2}} y\rho(x, y, z) dzdxdy$$

where $\rho(x, y, z) = z$. This triple integral can be evaluated as in part (b) above.

(9) Determine whether each of the following vector fields is conservative or not. If it is conservative then find a potential function for the vector field.

(a) $3(x^2 + y^2)^{3/2}(x\mathbf{i} + y\mathbf{j})$.
Solution: Let M, N denote the first and second components respectively of this vector field. Then

$$M_y = 9xy(x^2 + y^2)^{1/2}$$
$$N_x = 9xy(x^2 + y^2)^{1/2} \quad .$$

Hence $M_y = N_x$ holds on the whole plane, so the force field is conservative. A potential function is $(3/5)(x^2 + y^2)^{5/2}$.

(b) $sin(x)\mathbf{i} + y^2\mathbf{j}$.
Solution: If M, N denote the first and second components of this vector field then

$$M_y = 0$$
$$N_x = 0 \quad .$$

Hence $M_y = N_x$ on the whole plane, so this vector field is conservative. A potential function is $-cos(x) + y^3/3$.

(c) $y^2\mathbf{i} + x^4\mathbf{j}$.
Solution: We have that

$$M_y = 2y$$
$$N_x = 4x^3 \quad .$$

Thus $M_y \neq N_x$, so the vector field is not conservative.

(d) $(xy^2 - y)\mathbf{i} + (x^2y - x)\mathbf{j}$.
Solution: We have that

$$M_y = 2xy - 1$$
$$N_x = 2xy - 1 \quad .$$

Thus $M_y = N_x$ on the whole plane, so this vector field is conservative. A potential function is $x^2y^2/2 - xy$.

(e) $y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$.
Solution: Let M, N, P denote the first,second and third components respectively of this vector field. Then we have that the equalities

$$M_y = 2yz^3 = N_x$$

$$M_z = 3y^2z^2 = P_x$$
$$N_z = 6xyz^2 = P_y$$

hold on the entire plane, so this vector field is conservative. A potential func tion is xy^2z^3.

(f) $e^z y\mathbf{i} + e^z x\mathbf{j} + e^z xy\mathbf{k}$.
Solution: Let M, N, P denote the 3 components of this vector field. Note that the equalities

$$M_y = e^z = N_x$$
$$M_z = e^z y = P_x$$
$$N_z = e^z x = P_y$$

holds on the entire plane, so the vector field is conservative. A potential function is $e^z xy$.

(10) Find the total mass of the wire

$$\mathbf{r}(t) = t^3\mathbf{i} - 3t\mathbf{j} + t\mathbf{k}, \quad 1 \le t \le 4$$

with density given by $\rho(x, y, z) = x$. (Note: I have changed the density function and the vector valued function $\mathbf{r}(t)$.)
Solution: $mass = \int_c \rho(x, y, z)ds = \int_1^4 \rho(\mathbf{r}(t)) \parallel \mathbf{r}'(t) \parallel dt = \int_1^4 t^3(9t^4 + 10)^{1/2}dt = (9t^4 + 10)^{3/2}/54 \parallel_1^4$.

(11) Find the amount of work done by the force field $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$ on a particle moving along the path $\mathbf{r}(t) = 2t\mathbf{i} - t^2\mathbf{j}$, $0 \le t \le \pi$. (Note: I have changed the equation of the path.)
Solution: Work$=\int_C \mathbf{F} \bullet d\mathbf{r} = \int_0^\pi \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)dt = \int_0^\pi < 4t^2, 2t^3 > \bullet < 2, -2t > dt = \int_0^\pi 8t^2 - 4t^4 dt = (8t^3/3 - 4t^5/5) \parallel_0^\pi$.

(12) Consider the force field $\mathbf{F}(x, y) = sin(xy)\mathbf{i} + ((x/y)sin(xy) + cos(xy)/y^2)\mathbf{j}$. Using the fact that $\mathbf{F}$ is a conservative vector field, compute the work done by $\mathbf{F}$ as a particle moves along the path

$$\mathbf{r}(t) = t\mathbf{i} + 2^t\mathbf{j}, \quad 0 \le t \le 2.$$

Solution: A potential function is $f(x, y) = -cos(xy)/y$. so the work done is equal to

$$f(\mathbf{r}(2)) - f(\mathbf{r}(0)) = f(2, 4) - f(0, 1) = -cos(8)/4 - (-cos(0)) \quad .$$

(13) Use Green's Theorem to aid in the computation of $\int_C \mathbf{F}d\mathbf{r}$, where C is the curve traced out by the vector valued function

$$\mathbf{r}(t) = cos(t)\mathbf{i} + sin(t)\mathbf{j}, \quad -\pi/2 \le t \le \pi/2,$$

and where

$$\mathbf{F}(x, y) = xy\mathbf{i} + (y^3 + x)\mathbf{j} \quad .$$

Solution: Another curve C^* is traced out by the vector valued function

$$\mathbf{s}(t) = (1 - t)\mathbf{j}, \quad 0 \le t \le 2.$$

Let R be the region in the plane whose boundary is the union of the two curves C, C^*. By Greens Theorem we have

$$\int\int_R N_x - M_y dA = \int_C \mathbf{F} \bullet d\mathbf{r} + \int_{C^*} \mathbf{F} \bullet d\mathbf{s} \quad .$$

Since $N_x - M_y = 1 - x$ the double integral on the right in the above equality is equal (in polar coordinates) to $\int_{-\pi/2}^{\pi/2} \int_0^1 (1 - rcos(\theta))rdrd\theta = \int_{-\pi/2}^{\pi/2}((r^2/2 - r^3cos\theta)/3) \|_0^1)d\theta = \int_{-\pi/2}^{\pi/2} 1/2 - cos(\theta)/3d\theta = (\theta/2 - sin(\theta)/3) \|_{-\pi/2}^{\pi/2} = \pi/2 - 2/3$.
We also have the computation $\int_{C^*} \mathbf{F} \bullet d\mathbf{s} = \int_0^2 \mathbf{F}(\mathbf{s}(t)) \bullet \mathbf{s}'(t)dt =$
$\int_0^2 < 0, (1-t)^3 > \bullet < 0, -1 > dt = \int_0^2 -(1-t)^3 dt = (1-t)^4/4 \|_0^2 = 0$. Thus $\int_C \mathbf{F} \bullet d\mathbf{r} = \int\int_R N_x - M_y dA = \pi/2 - 2/3$.