MIDTERM 203; SPRING 2009

Instructions: Fill in the blank lines on this sheet. Do any 4 of the following 5 problems in the spaces provided; do not do all 5 problems. Be sure to show some work, or give an explanation for each of your answers.

Print Name:

Write ID number:

(1) Set $f(x, y, z) = y^{-3}cos^2(xz)$.

(a) What is the domain of f?
Solution: Domain(f)=$\{(x, y, z) : x, y, z \in R, y \neq 0\}$.

(b) Compute the partial derivatives $f_x(x, y, z)$, $f_{yy}(x, y, z)$.
Solution:

$$f_x(x, y, z) = -2zy^{-3}cos(xz)sin(xz)$$
$$f_{yy} = 12y^{-5}cos^2(xz)$$

(2) Let $g(x,y)$ denote a function of 2 variables which satisfies

$$g(1,1) = -4$$
$$\bigtriangledown g(1,1) = -2\mathbf{i} + \mathbf{j}.$$

Consider the surface in 3 space defined by $g(x,y)+z^2 = 0$. Find the equation for the tangent plane to this surface at the point (1,1,2) on the surface.

Solution: First note that since $\bigtriangledown g(1,1) = g_x(1,1)\mathbf{i} + g_y(1,1)\mathbf{j}$ it follows that

$$(i) \quad g_x(1,1) = -2, g_y(1,1) = 1.$$

Set $f(x,y,z) = g(x,y) + z^2$; then

$$(ii) \quad f_x(x,y,z) = g_x(x,y), f_y(x,y,z) = g_y(x,y), f_z(x,y,z) = 2z.$$

Moreover the equation for the given surface can be written as $f(x,y,z) = 0$. Thus the equation for the desired tangent plane to this surface at $(1,1,2)$ is

$$(iii) \quad f_x(1,1,2)(x-1) + f_y(1,1,2)(y-1) + f_z(x,y,z)(z-2) = 0.$$

Now, combining (i)-(iii) above we see that the equation for the tangent plane is

$$-2(x-1) + (y-1) + 4(z-2).$$

(3) Let $\mathbf{r}(t)$ denote the position of a moving partical in 3-dimensional space at time t which passes thru the point $\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ at time $t = 1$. Suppose that the acceleration and velocity of the partical — denoted by $\mathbf{a}(t), \mathbf{v}(t)$ — satisfy

$$\mathbf{a}(t) = -16cos(\pi t)\mathbf{j}$$
$$\mathbf{v}(1) = -2\mathbf{j} - \mathbf{k}.$$

Find the position of the partical at time $t = 2$.
Solution: $\mathbf{v}(t)$ is an antiderivative for $\mathbf{a}(t) = -16cos(\pi t)\mathbf{j}$; thus

$$(i) \quad \mathbf{v}(t) = -\frac{16}{\pi} sin(\pi t)\mathbf{j} + \mathbf{u}$$

for some vector $\mathbf{u}$. Using this formula for $t = 1$ we get that

$$(ii) \quad \mathbf{v}(1) = \mathbf{u};$$

so by combing (ii) with the hypothesis of this problem (that $\mathbf{v}(1) = -2\mathbf{j} - \mathbf{k}$) we get

$$(iii) \quad \mathbf{u} = -2\mathbf{j} - \mathbf{k}.$$

Now (i) and (iii) imply that

$$(iv) \quad \mathbf{v}(t) = (-\frac{16}{\pi} sin(\pi t) - 2)\mathbf{j} - \mathbf{k}.$$

Note also that $\mathbf{r}(t)$ is an antiderivative for $\mathbf{v}(t)$; thus by (iv) we get that

$$(v) \quad \mathbf{r}(t) = (\frac{16}{\pi^2} cos(\pi t) - 2t)\mathbf{j} - t\mathbf{k} + \mathbf{w}$$

for some vector $\mathbf{w}$. We can solve for $\mathbf{w}$ by using (v) to compute $\mathbf{r}(1)$ and comparing it to the hypothesis for this problem (that $\mathbf{r}(1) = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$). Thus we get that

$$(vi) \quad \mathbf{w} = \mathbf{i} + (\frac{16}{\pi^2} + 1)\mathbf{j} - \mathbf{k}.$$

Now combining (v) and (vi) we get that

$$(vii) \quad \mathbf{r}(t) = \mathbf{i} + (\frac{16}{\pi^2} cos(\pi t) - 2t + \frac{16}{\pi^2} + 1)\mathbf{j} - (t + 1)\mathbf{k}.$$

To complete this problem use (vii) to compute $\mathbf{r}(2)$.

(4) Set $f(x,y) = xsin(y)$.

(a) Find all critical points for $f(x,y)$.
Solution: Since $f_x(x,y) = sin(y)$ and $f_y(x,y) = xcos(y)$, the equations which define the critical points for f are

$$sin(y) = 0$$
$$xcos(y) = 0.$$

The solutions to these two equations are

$$(x,y) = (0, n\pi),$$

where n is equal to any integer.

(b) Use the second partial derivative test to determine the nature of each critical point for $f(x,y)$.
Solution: $f_{xx}(x,y) = 0$, $f_{xy}(x,y) = cos(y)$, $f_{yy}(x,y) = -xsin(y)$. Thus

$$d = f_{xx}(0,n\pi)f_{yy}(0,n\pi) - (f_{xy}(0,n\pi))^2 = -(cos(n\pi))^2 = -1.$$

So each of the critical points $(0, n\pi)$ is a saddle point.

(5) Let $f(x, y, z)$ denote a function of 3-variables which satisfies

$$\bigtriangledown f(0,1,2) = \mathbf{i} - \mathbf{j} + 3\mathbf{k}.$$

Compute the directional derivative $D_{\mathbf{u}} f(0,1,2)$ where $\mathbf{u}$ is a unit vector defined by

$$\mathbf{u} = sin(1)\mathbf{j} - cos(1)\mathbf{k}.$$

(Express your answer in terms of $sin(1), cos(1)$.)
Solution: Use the formula

$$D_{\mathbf{u}} f(0,1,2) = \bigtriangledown f(0,1,2) * \mathbf{u},$$

where * denotes dot product.