(1) Set \( f(x) = x^3 - 3x - 2 \).

(a) On which intervals is \( f(x) \) increasing and on which intervals is it decreasing?

**Solution:** \( f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1) \).

\( f'(x) > 0 \) if both factors \((x + 1), (x - 1)\) are positive (i.e. \( x > 1 \)) or if both factors \((x + 1), (x - 1)\) are negative (i.e. \( x < -1 \)). Thus \( f(x) \) is increasing on the intervals \((-\infty, -1), (1, \infty)\).

\( f(x) < 0 \) if the two factors \((x + 1), (x - 1)\) have opposite signs (i.e. \(-1 < x < 1\)). Thus \( f(x) \) is decreasing on \((-1, 1)\).

(b) Over which interval is the graph of \( f(x) \) concave up and over which interval is it concave down?

**Solution:** \( f''(x) = 6x \). Thus \( f''(x) \) is positive (or negative) on the interval \((0, \infty)\) (or the interval \((-\infty, 0))\). So the graph of \( f(x) \) is concave up over the interval \((0, \infty)\) and is concave down over the interval \((-\infty, 0)\).

(c) Sketch the graph for \( f(x) \) clearly indicating its y-intercept, where it is increasing and decreasing, and where it is concave up and concave down.

**Solution:** The y-intercept is \((0, -2)\); the points \((-1, 0)\) and \((1, -4)\) are also on the graph.

(2) Let \( f(x), g(x) \) denote real valued functions which are defined and differentiable on the whole real number line; and set \( h(x) = f(g(x)) \). If \( f(3) = 5, f'(3) = -1 \) and \( g(8) = 3, g'(8) = -2 \), then find an equation for the tangent line to the graph of \( h(x) \) when \( x = 8 \).

**Solution:** Let \( T_8 \) denote the tangent line to the graph of \( h(x) \) at \( x = 8 \). Note that \( h(8) = f(g(8)) = f(3) = 5 \); thus the point \((8, 5)\) is on \( T_8 \). Note also that the slope of \( T_8 \) is equal to \( h'(8) \), and by the chain rule we have that \( h'(8) = f'(g(8))g'(8) = f'(3)(-2) = (-1)(-2) = 2 \); thus the slope of \( T_8 \) is equal to \( 2 \).

Since \( T_8 \) contains \((8, 5)\) and has slope equal \( 2 \), the (point slope) equation for \( T_8 \) is

\[
\frac{y - 5}{x - 8} = 2.
\]
(3) Compute the following limit:

\[
\lim_{x \to 0} \frac{e^x \tan(x)}{x} = ?
\]

**Solution:** Since \( \tan(0) = 0 \) the above limit is equal to

\[
\lim_{x \to 0} \frac{e^x \tan(x) - e^0 \tan(0)}{x - 0},
\]

and this last limit is (by definition of the derivative at \( x = 0 \)) equal to \( f''(0) \) where \( f(x) = e^x \tan(x) \). By the product rule for derivatives we have

\[
f'(x) = e^x \tan(x) + e^x \sec^2(x);
\]

thus \( f'(0) = e^0 \tan(0) + e^0 \sec^2(0) = 0 + 1 = 1 \).

Thus the original limit is equal to 1.

(4) Compute the \( n \)'th derivative of the function \( f(x) \) for each of the choices for \( n \) and for \( f(x) \) given below.

(a) \( n = 1 \) and \( f(x) = \frac{\sin^{-1}(x)}{\sin(x)} \).

**Solution:** Using the quotient rule for derivatives we have that

\[
f^{(1)}(x) = \frac{(\sin^{-1}(x))(\sin(x)) - \sin^{-1}(x)(\sin(x))}{\sin^2(x)} = \frac{\sin(x)}{\sqrt{1-x^2}} \frac{\sin^{-1}(x) \cos(x)}{\sin^2(x)}.
\]

(b) \( n = 22 \) and \( f(x) = \cos(x) + e^x \).

**Solution:** The for a positive integer \( k \) the \( k \)-th derivative \( (\cos(x))^k \) is equal to \( -\sin(x), -\cos(x), \sin(x), \cos(x) \) depending on whether \( k \) is equal to 1, 2, 3, 0 mod 4 respectively. So \( (\cos(x))^{(22)} = -\cos(x) \).

For any positive integer \( k \) the \( k \)-th derivative \( (e^x)^k \) is equal to \( e^x \); so \( (e^x)^{(22)} = e^x \).

Thus

\[(\cos(x) + e^x)^{(22)} = (\cos(x))^{(22)} + (e^x)^{(22)} = -\cos(x) + e^x\]

(5) Find an example of a polynomial \( f(x) \) of degree 3 which satisfies the following properties:

\[
\begin{align*}
f(0) &= 1 \\
f^{(1)}(0) &= 2 \\
f^{(2)}(0) &= 3 \\
f^{(3)}(0) &= 4
\end{align*}
\]

(Here \( f^{(n)}(x) \) denotes the \( n \)-th derivative of \( f(x) \).)

**Solution:** Note that if

\[
f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3,
\]

where the \( a_0, a_1, a_2, a_3 \) are the real number coefficients for the 3rd degree polynomial \( f(x) \), then

\[
\begin{align*}
f^{(1)}(x) &= a_1 + 2a_2 x + 3a_3 x^2 \\
f^{(2)}(x) &= 2a_2 + 6a_3 x
\end{align*}
\]
\[ f^{(3)}(x) = 6a_3. \]

If we set \( x = 0 \) in these last four equations we get
\[
\begin{align*}
 f(0) &= a_0 \\
 f^{(1)}(0) &= a_1 \\
 f^{(2)}(0) &= 2a_2 \\
 f^{(3)}(0) &= 6a_3.
\end{align*}
\]

Now comparing these last four equations to the four equalities given in the statement of problem #5 we can solve for the \( a_i \) as follows:
\[
\begin{align*}
 a_0 &= 1 \\
 a_1 &= 2 \\
 a_2 &= \frac{3}{2} \\
 a_3 &= \frac{2}{3}.
\end{align*}
\]

Thus
\[ f(x) = 1 + 2x + \frac{3}{2}x^2 + \frac{2}{3}x^3. \]