MIDTERM II; MAT 131 (SPRING, 08)

(1) Set $f(x) = x^3 - 3x - 2$.

(a) On which intervals is $f(x)$ increasing and on which intervals is it decreasing?
Solution: $f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$.
$f'(x) > 0$ if both factors $(x+1), (x-1)$ are positive (i.e. $x > 1$) or if both factors $(x+1), (x-1)$ are negative (i.e. $x < -1$). Thus $f(x)$ is increasing on the intervals $(-\infty, -1), (1, \infty)$.
$f(x) < 0$ if the two factors $(x+1), (x-1)$ have opposite signs (i.e. $-1 < x < 1$). Thus $f(x)$ is decreasing on $(-1, 1)$.

(b) Over which interval is the graph of $f(x)$ concave up and over which interval is it concave down?
Solution: $f''(x) = 6x$. Thus $f''(x)$ is positive (or negative) on the interval $(0, \infty)$ (or the interval $(-\infty, 0)$). So the graph of $f(x)$ is concave up over the interval $(0, \infty)$ and is concave down over the interval $(-\infty, 0)$.

(c) Sketch the graph for $f(x)$ clearly indicating its y-intercept, where it is increasing and decreasing, and where it is concave up and cancave down.
Solution: The y-intercept is $(0, -2)$; the points $(-1, 0)$ and $(1, -4)$ are also on the graph.

(2) Let $f(x), g(x)$ denote real valued functions which are defined and differentiable on the whole real number line; and set $h(x) = f(g(x))$. If $f(3) = 5, f'(3) = -1$ and $g(8) = 3, g'(8) = -2$, then find an equation for the tangent line to the graph of $h(x)$ when $x = 8$.
Solution: Let T_8 denote the tangent line to the graph of $h(x)$ at $x = 8$. Note that $h(8) = f(g(8)) = f(3) = 5$; thus the point $(8, 5)$ is on T_8. Note also that the slope of T_8 is equal to $h'(8)$, and by the chain rule we have that $h'(8) = f'(g(8))g'(8) = f'(3)(-2) = (-1)(-2) = 2$; thus the slope of T_8 is equal to 2.

Since T_8 contains $(8, 5)$ and has slope equal 2, the (point slope) equation for T_8 is

$$\frac{y-5}{x-8} = 2.$$

(3) Compute the following limit:

$$limit_{x \to 0} \frac{e^x tan(x)}{x} = ?$$

Solution: Since $tan(0) = 0$ the above limit is equal to

$$limit_{x \to 0} \frac{e^x tan(x) - e^0 tan(0)}{x - 0},$$

and this last limit is (by definition of the derivative at $x = 0$) equal to $f'(0)$ where $f(x) = e^x tan(x)$. By the product rule for derivatives we have $f'(x) = e^x tan(x) + e^x sec^2(x)$; thus $f'(0) = e^0 tan(0) + e^0 sec^2(0) = 0 + 1 = 1$.
Thus the original limit is equal to 1.

(4) Compute the n'th derivative of the function $f(x)$ for each of the choices for n and for $f(x)$ given below.

(a) $n = 1$ and $f(x) = \frac{sin^{-1}(x)}{sin(x)}$.
Solution: Using the quotient rule for derivatives we have that $f^{(1)}(x) = \frac{(sin^{-1}(x))^{(1)} sin(x) - sin^{-1}(x)(sin(x))^{(1)}}{sin^2(x)} = \frac{\frac{sin(x)}{\sqrt{1-x^2}} - sin^{-1}(x)cos(x)}{sin^2(x)}$.

(b) $n = 22$ and $f(x) = cos(x) + e^x$.
Solution: The for a positive integer k the k-th derivative $(cos(x))^{(k)}$ is equal to $-sin(x), -cos(x), sin(x), cos(x)$ depending on whether k is equal to $1, 2, 3, 0$ mod 4 respectively. So $(cos(x))^{(22)} = -cos(x)$.
For any positive integer k the k-th derivative $(e^x)^{(k)}$ is equal to e^x; so $(e^x)^{(22)} = e^x$.
Thus

$$(cos(x) + e^x)^{(22)} = (cos(x))^{(22)} + (e^x)^{(22)} = -cos(x) + e^x.$$

(5) Find an example of a polynomial $f(x)$ of degree 3 which satisfies the following properties:

$$f(0) = 1$$
$$f^{(1)}(0) = 2$$
$$f^{(2)}(0) = 3$$
$$f^{(3)}(0) = 4$$

(Here $f^{(n)}(x)$ denotes the n'th derivative of $f(x)$.)
Solution: Note that if

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3,$$

where the a_0, a_1, a_2, a_3 are the real number coefficients for the 3rd degree polynomial $f(x)$, then

$$f^{(1)}(x) = a_1 + 2a_2 x + 3a_3 x^2$$
$$f^{(2)}(x) = 2a_2 + 6a_3 x$$

$$f^{(3)}(x) = 6a_3.$$

If we set $x = 0$ in these last four equations we get

$$f(0) = a_0$$
$$f^{(1)}(0) = a_1$$
$$f^{(2)}(0) = 2a_2$$
$$f^{(3)}(0) = 6a_3.$$

Now comparing these last four equations to the four equalities given in the statement of problem #5 we can solve for the a_i as follows:

$$a_0 = 1$$
$$a_1 = 2$$
$$a_2 = \frac{3}{2}$$
$$a_3 = \frac{2}{3}.$$

Thus

$$f(x) = 1 + 2x + \frac{3}{2}x^2 + \frac{2}{3}x^3.$$