SOME SUGGESTED HW SOLUTIONS

1. HW 8

- Problem 42.4

$$\begin{aligned}\int_0^{\pi} e^{(1+i)x}dx &= \frac{1}{1+i}\left(e^{(1+i)x}\Big|_0^{\pi}\right)\\ &= \frac{1}{1+i}\left(e^{\pi}e^{\pi i} - e^0\right)\\ &= \frac{1}{1+i}\left(-e^{\pi} - 1\right)\\ &= \frac{(1-i)(-e^{\pi}-1)}{2}\end{aligned}$$

Then note that the real part of this solution is $\frac{(-e^{\pi}-1)}{2}$ and that the imaginary part of this solution is $\frac{(e^{\pi}+1)}{2}$

- Problem 47.2

We want an upper bound on $\left|\int_{\mathcal{C}} \frac{dz}{z^4}\right|$, where $\mathcal{C}$ is the given oriented contour, the line segment connecting i to 1.

We will use the upper bound that is the product of the maximum value of $|\frac{1}{z^4}|$ on $\mathcal{C}$ and the length of $\mathcal{C}$. The maximum value of the modulus of a complex number z on the contour is 1, and the minimum value is $\frac{1}{\sqrt{2}}$. The least value for $|z|$ will give us the greatest value for $|\frac{1}{z^4}|$. So we have that $|\frac{1}{z^4}| \leq \frac{1}{(\frac{1}{\sqrt{2}})^4} = 4$.

These give us $\left|\int_{\mathcal{C}} \frac{dz}{z^4}\right| \leq 4 \cdot \sqrt{2}$.

Date: April 10, 2018.

- Problem 47.4

We want a bound on
$$\left|\int_{\mathcal{C}_R} \frac{2z^2-1}{z^4+5z^2+4}dz\right|,$$
where $\mathcal{C}_R$ is the top half of a positively oriented semicircle of radius R, centered at the origin.

We have $|z^4+5z^2+4| = |z^2+4||z^2+1|$.

From the triangle inequality, $|z^2+4| \geq ||z^2|-|4|| = |R^2-4|$.

And similarly, $|z^2+1| \geq |R^2-1|$.

Also from the triangle inequality, we have $|2z^2-1| \leq 2|z^2|+|1| = 2R^2+1$.

Combining these inequalities, and noting that the length of $\mathcal{C}_R$ is πR, gives us the desired
$$\left|\int_{\mathcal{C}_R} \frac{2z^2-1}{z^4+5z^2+4}dz\right| \leq \frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)}.$$

Then take the limit of this quotient as $R \to \infty$.

2. HW9

- Problem 49.4

We are looking at the function $f_2(z) = z^{1/2} = e^{\frac{1}{2}\log z}$, and we are using the branch of the log function with $\frac{\pi}{2} < \theta < \frac{5\pi}{2}$. Note that this branch is indeed analytic in a connected open set that contains any contour from -3 to 3 running below the real axis, since it only fails to be analytic along the branch cut, which is the non-negative imaginary axis. So $f_2(z)$ has an antiderivative throughout this domain, and we can use the antiderivative to evaluate the definite integral.

We have:

$\int_{\mathcal{C}_2} z^{\frac{1}{2}}dz = \int_{-3}^{3} f_2(z)dz = \frac{2}{3}r\sqrt{r}e^{\frac{3\theta i}{2}}\Big|_{z=-3}^{z=3}$

In the final step we are using our choice of branch, with $\frac{\pi}{2} < \theta < \frac{5\pi}{2}$, to determine the correct choice of argument for the points $z=-3$ and $z=3$. For $z=-3$ we have $\theta=\pi$ and for $z=3$ we have $\theta=2\pi$. [Compare this to the example in section 48.] Continuing we have:

$\int_{\mathcal{C}_2} z^{\frac{1}{2}}dz = \frac{2}{3}3\sqrt{3}(e^{\frac{6\pi i}{2}} - e^{\frac{3\pi i}{2}}) = 2\sqrt{3}(-1+i).$

- Problem 57.4

We are determining the value of $g(z) = \int_{\mathcal{C}} \frac{s^3+2s}{(s-z)^3} ds$ in the case where z lies inside of $\mathcal{C}$ and in the case where z lies outside of $\mathcal{C}$, where $\mathcal{C}$ denotes an simple closed contour in the plane with positive orientation.

(1) Suppose first that the point z is inside of $\mathcal{C}$. Then, since the polynomial s^3+2s is entire, we can use the generalization of the Cauchy Integral Formula. We have $f^{''}(s) = 6s$. I evaluate this at any fixed point z inside of the contour, obtaining $f^{''}(z) = 6z$. Then

$$\int_{\mathcal{C}} \frac{s^3+2s}{(s-z)^3} ds = \frac{(6z)(2\pi i)}{2!} = 6\pi i z.$$

(2) Suppose next that the point z lies outside of $\mathcal{C}$. In this lovely case, the function $\frac{s^3+2s}{(s-z)^3}$ is analytic on and inside of $\mathcal{C}$. (Its only singularity is at the point $s = z$, which is assumed outside of this region.) So the Cauchy-Goursat Theorem tells us that

$$\int_{\mathcal{C}} \frac{s^3+2s}{(s-z)^3} ds = 0.$$

- Problem 59.8

For part a, just do the algebra for the verification. If you find the notation cumbersome, try first fixing some small k, try $k = 3$, and first do the algebra in this case.

For part b:

$$P(z) = a_0 + a_1 z + a_2 z + \cdots + a_n z^n$$
$$P(z_0) = a_0 + a_1 z_0 + a_2 z_0^2 + \cdots + a_n z_o^n$$

Subtracting we have $P(z) - P(z_0) = a_1(z - z_0) + a_2(z^2 - z_0^2) + \cdots + a_n(z^n - z_0^n)$

Next use the result you verified in part a. So you will have:

$$P(z) = P(z) - 0 = P(z) - P(z_0) = (z - z_0)(a_1 + a_2(z + z_0) + \cdots + a_n(z^{n-1} + \cdots z_0^{n-1}).$$

Explain why this is the desired result. (Identify $Q(z)$ in the equalities above.)