One goal for this course is for you to develop your skill in effectively communicating mathematics. With this in mind, you should clearly write up your solutions.

(1) (a) Give a clear, concise definition of an equivalence relation on a set.
(b) In class we define the rational numbers as:

\[ \mathbb{Q} := \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0, \text{ and } \frac{a}{b} \sim \frac{c}{d} \text{ if } ad = bc \right\} \]

Prove that the relation \( \{\sim\} \), in this definition, is an equivalence relation on the set of ratios of integers with non-zero denominator.

(2) We have been developing an analogy between the ring of integers and the ring of polynomials with rational coefficients.

(a) Write a paragraph explaining your current understanding of this analogy.
(b) Prove that the Euclidean Algorithm produces a greatest common divisor of two integers.

(3) Rational functions may be viewed as functions whose domain is a subset of \( \mathbb{R} \). They also have an algebraic structure, and this exists independent of the functional definition.

(a) Write our definition of the field of rational functions as a set of equivalence classes of ratios of polynomials.

(b) Take a look at the rational functions given below and, using our definition, group them into equivalence classes. Explain your solution.

\[ \frac{1}{x}, \frac{x+1}{x-1}, \frac{x^2-1}{x^2-2x+1}, \frac{x^2}{x+1}, \frac{x-1}{x^2-x}, \frac{x^2+x}{x^2-x} \]

(c) Now think of the rational functions as actual real-valued functions. What does it mean, in terms of their functional representation, for two rational functions to lie in the same equivalence class.
(4) For each item below, write an equation for a rational function that meets the given criteria. Sketch the graph of each function on labelled coordinate axes.

(a) \( \lim_{x \to \infty} f(x) = 0 \) and the range of \( f(x) \) includes only positive numbers.

(b) \( \lim_{x \to \infty} g(x) = -\infty, \lim_{x \to -\infty} g(x) = \infty \), and the function has no asymptotes.

(c) \( \lim_{x \to 5^+} h(x) = \infty \) and \( \lim_{x \to 5^-} h(x) = \infty \).

(d) The graph of the rational function has at least three disjoint components.

(5) More may be posted by 5 p.m. on Friday, March 1.