SOME NOTES FOR THE FINAL EXAM

The following sections of the textbook were covered in class or in readings: 1-26, 28-40, excluding inverse hyperbolic functions, 41-59, 60-66, 68, 72–(results mentioned in class and assumed in HW)–, 74-77.

Below some important topics are discussed in more detail, with some suggested problems or examples.

- Complex numbers–very basic arithmetic.

  You should know basic arithmetic of complex numbers and demonstrate facility in working with both rectangular and polar forms of complex numbers; you should be able to translate among polar form, rectangular form, and geometric representation of a complex number.

  You should understand addition, multiplication, and conjugation arithmetically and geometrically. What happens geometrically when complex numbers are added or multiplied? Understand the geometry of the power function z^n. Look at specific examples.

  Look back at the problems involving the power function on each exam, and make sure you can solve these.

  Find all complex roots of the polynomials $z^5 - 1$ and $z^3 + 8$. Find the roots algebraically and graph the roots in the complex plane. Write each root in the form $c_0 e^{\frac{2k\pi i}{n}}$ for appropriate n, k, and principal root c_0. If you need to, work through the examples in section 11 and some of the related exercises.

- Describe the image of a mapping of a subset of the complex plane.

- Basic topology of the complex plane. Open set, closed set, bounded set, boundary point, interior point, accumulation point, neighborhood, deleted neighborhood, connected set, Domain.

- Limits and continuity

  Know and understand the definition of the limit of a function at a point and use theorems on limits to compute limits.

  Understand and use the main theorems on limits involving the point at infinity. (There was a problem from exam 1 you might want to re-do, along with exercises from section 18.)

  Continuity

- Some elementary functions

  You should know (and be able to use) definitions for the complex exponential function, trig functions, power function, and logarithmic function, (Log, log, and various branches), along with

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polynomial functions and rational functions.

You should demonstrate understanding that the definitions for complex functions *extend* our definitions of the similarly-named real-valued functions. (When we restrict the complex function to the real line, we get our familiar real-valued function.)

The exponential function is periodic. What is the period? How is periodicity of the exponential function related to the fact that the logarithm is a multi-valued function?

- Differentiability and analyticity

  What is the definition of the derivative of a complex function at a point. How do conditions for, or results about, differentiability of a complex function compare to those for real-valued functions of a real variable.

  Rules for differentiation, (section 20.)

  Cauchy-Riemann equations. Relation to necessary conditions for differentiability and sufficient conditions for differentiability. Here, it would be useful to understand example 3 from page 65 (going back to section 20 as suggested) and to contrast this to the statement of the main theorem on page 66. Work out some examples for which you can use the main theorem to prove differentiability. Work out examples for which you can prove a function is not differentiable at a point by showing that the Cauchy-Riemann equations do not hold.

  It would be useful to know the rectangular and also polar coordinate versions of the Cauchy-Riemann equations.

  Analytic functions. Properties of analytic functions and theorems based on analyticity.

  Reflection principal. We initially presented this as an idea related to even and odd functions in the real setting. If you work through problem #5 on page 85, this would help you to:

  - Review the reflection principal, and
  - Work through a proof involving the Cauchy-Riemann equations.

- Multi-valued functions.

  Part of the definition of a real valued function of a real variable is the fact that an element in the domain maps to exactly one element in the range. In the study of complex functions, we allow functions to be multi-valued. Our first example of a multi-valued function came in the study of roots of a complex number, which is in section 10, where the principal root of a complex function was also defined. (In section 9 the *argument* and *principal argument* of a complex number are defined.)

  For a multi-valued function, you should be able to identify all values of the function evaluated at a single complex number, regardless of whether there are infinitely many or finitely many values.

  You should understand how the restriction to a single branch makes the function single-valued.

Understand how the branch cut affects continuity (and hence analyticity) of a function.

Explain how a Riemann surface for a multi-valued function is used to make the function single-valued. How are the Riemann surfaces for the log function and for the root function(s) created by gluing along branch cuts?

You should be able to compute contour integrals involving branch cuts. In particular, you should be able to work with situations in which an antiderivative is not defined throughout the domain in which the contour lies.

Some of the main ideas are illustrated in sections 9-10, 31-34, 35-36, 46, and 48. It would be useful to re-do Exam 1 problem 10 and to use the methods illustrated in section 48 to solve problem 3 from exam 2.

- Sequences and series

Sequences of complex numbers, convergent sequences.

Series, convergent series, understanding of the sum of a convergent series as a limit of a sequence of partial sums.

Conditions for which a complex-valued function has a convergent Taylor series. Domain of convergence.

Conditions for which a complex-valued function has a convergent Laurent series. Understanding of the domain of convergence.

Use formulas for known Taylor series to determine Taylor series or Laurent series for other functions.

You should know, at least, the Taylor series for e^z, $\sin z$, $\cos z$, and $\frac{1}{1-z}$, and their respective domains of convergence, (although you could derive them if needed).

Describe region of convergence for Taylor series and Laurent series. It would be useful to completely understand example 2 on page 203 and HW problem 68.5

- Integration

Derivatives and integrals of *complex-valued* function of a *real* variable. (Sections 41-42.)

Contours. Determine a parameterization of a given contour and determine the contour from a given parameterization.

Definition of a contour integral. (Section 44)

Computing contour integrals using parameterizations and using the theory developed in the second half of chapter 4.

It is important to understand the statements of the main theorems on integration and how (when) to use them. Try (re)-doing the computational problems from this chapter, with attention to the theory you apply in solving each problem.

Determining an upper bound on the modulus of a contour integral. You should be able to do some simple problems involving bounding the modulus of an integral. Finding (and re-reading) some of the proofs in which this method of determining an upper bound was used could help.

- Residues

  What is an isolated singular point of a function? Give some examples

  Define the residue of a function at an isolated singular point.

  Compute the residue of a given function at a given point.

  Use theory of residues to compute contour integrals.

- Some main ideas

  - Write three examples that illustrate how the theory of functions of a complex variable is different from that of functions of a real variable. You could, for example, consider theorems that hold in one setting for which analogous statements fail to hold in the other setting. Think, for example, about continuity, differentiability, and integration theory.

  - Write a couple of examples to illustrate ideas or results that easily extend from the theory of real-valued functions of a real variable to complex function theory.