MATH 301/501 FINAL EXAM, FALL 2020
DUE AT THE END OF THE DAY ON TUESDAY, DECEMBER 15.

One goal for this course is for you to develop your skill in effectively communicating mathematics. With this in mind, you should clearly write up your solutions. A solution with little or no justification will receive little or no credit.

You can work with other students in the class, and you can use your class notes. Using any other outside material, including electronic resources, is a violation of the University policy of academic integrity. Discussing the problems with anyone outside of class is a violation of the University policy of academic integrity. You are welcome to work with other students in this class; the solutions you submit should be your own writing and communicate your own understanding.

(1) In class we studied determined conditions on coprime integers a, b, and c that satisify the equation $a^2 + b^2 = c^2$. We determined, for example, that the c is odd and that a and b are of opposite parity. Consider instead the equation

$$x^2 + y^2 = p,$$

where p is a prime. That is, consider conditions for which an odd prime number can be expressed as the sum of two squares. Study this question, and see what you can prove about p, x and y. You should probably start–as we did with the Pythagorean triples–by constructing some explicit examples. Carefully explain any patterns you see and try to prove any hypothesis you make.

(2) Read the following Regents exam problem:

Which expression is always equivalent to $\sin x$ when $0 < x < \frac{\pi}{2}$?

(a) $\cos\left(\frac{\pi}{2} - x\right)$
(b) $\cos\left(\frac{\pi}{4} - x\right)$
(c) $\cos 2x$
(d) $\cos x$

(i) Give your solution to this problem.

(ii) *Prove* that your solution is correct. Clearly state the definition of the sine and cosine functions that you use in your proof.

(iii) Sketch a graph on clearly labeled coordinate axes to illustrate this problem.

(iv) The cofunction identity assessed in this exam problem is one that you will want your students to understand. Write 1-2 paragraphs explaining how you might develop conceptual understanding of this identity with your students.

(3) Suppose $(f \circ g)(x) = e^{\frac{-x^2}{2}}$.

(a) Find f and g.

(b) Find the function $g \circ f$, (for your g and f from part (a).)

(c) Are f and g injective? Explain.

(d) Is $f \circ g$ an injective function? Explain.

(e) Is $g \circ f$ an injective function? Explain.

(4) Let z_1, z_2 and z_3 denote complex numbers that lie on the unit circle in the complex plane. We proved that: if these complex numbers are the vertices of an equilateral triangle, then the sum $z_1 + z_2 + z_3 = 0$. Prove the converse.

That is: For three complex numbers on the unit circle, if $z_1 + z_2 + z_3 = 0$, then the points are the vertices of an equilateral triangle.

(5) 25 **points** Consider the function f given by $f(x) = (x-1)(x^5 + x^4 + x^3 + x^2 + x + 1)$.

(a) Find all rational zeros. Justify your solution algebraically.

(b) Sketch a graph of the function, on a clearly labeled and scaled set of coordinate axes.

(c) Determine the exact value of all complex zeros. Justify your solution algebraically.

(d) Graph the set of complex zeros in the complex plane.

(6) (a) Consider the composition of reflections $r_{\ell_3} \circ r_{\ell_2} \circ r_{\ell_1}$, for the lines whose equation is given below. What is the equation of the line that defines the composite reflection? Prove your result.

$$\ell_1 : y - 2 = 2x$$
$$\ell_2 : y - 2x = 0$$
$$\ell_3 : 2x - y = 3$$

(b) Consider the composition of reflections $r_{\ell_3} \circ r_{\ell_2} \circ r_{\ell_1}$, for the lines whose equation is given below. What is the equation of the line that defines the composite reflection? Prove your result.

$$\ell_1 : y = 0$$
$$\ell_2 : x = 0$$
$$\ell_3 : y = x$$

(7) Take a look at the January 2020 geometry regents exam: `https://www.nysedregents.org/geometryre/`. Look over problems 15, 17, 22, and 25, all of which assess student understanding of isometries, and make sure you can solve them. Think about the conceptual understanding you want students to have in order to be able to solve these problems. Write 1 - 2 pages discussing the mathematical ideas with which you would engage your students. Describe some mathematical tasks, (more conceptual than those on the Regents exam), that would teach your student about isometries. Please don't write a lesson plan; I am only interested in your mathematical ideas for developing conceptual understanding with your students. This problem will be graded on how well you communicate your own mathematical understanding of isometries.

(8) For each rational function below, do the following:

- Determine vertical asymptotes, zeros, and end behavior, (including a description of horizontal or slant asymptotes).
- Sketch a graph on clearly labelled coordinate axes.
- Write a rational function that is equivalent to, but not equal to, the given function.

(a) $f(x) = \frac{4-9x^2}{3x+2}$

(b) $g(x) = \frac{2x-3}{7x+2}$

(c) $h(x) = \frac{x^3-8x^2+4x-32}{x^5-1}$