MAT 401 HOMEWORK–4 DUE IN CLASS, NOVEMBER 21

One goal for this course is for you to develop your skill in effectively communicating mathematics. With this in mind, you should clearly write up your solutions. Solutions with little or no justification will receive little or no credit.

(1) Please make sure you are reading each section, regardless of who is scheduled to present. You should have read all of chapter 3 and should be getting into chapter 4.

(2) In class we looked briefly at the field extension $\mathbb{Q}(\zeta_3)/\mathbb{Q}$. Students suggested that this is a field extension of degree 3 or of degree 2. Prove that this is a degree 2 extenstion, by giving an explicit $\mathbb{Q}$-linear dependence relation among 1, ζ_3, and ζ_3^2.

(3) Let p denote a prime number, (fix a specific one if you like), and define

$$\mathbb{Z}_{(p)} := \{\frac{a}{b} : a, b \in \mathbb{Z}, p \nmid b\}.$$

Define $\mathrm{ord}_p(n) := k$, where $n = p^k u$, where $p^k | n$ and $p^{k+1} \nmid n$.

(a) Prove that the ring $\mathbb{Z}_{(p)}$ is a local ring with unique maximal ideal (p).

(b) Prove that the ord function, as defined above, is a discrete valuation on the ring $\mathbb{Z}_{(p)}$.

(c) In the inequality $\mathrm{ord}_p(mn) \geq \min\{\mathrm{ord}_p(m), \mathrm{ord}_p(n)\}$, give an explicit example for which this is a strict inequality and another for which the equality holds.

(4) In his presentation, Connor was able to quickly justify the results in example 3.16. Work out the details to prove the claims made by the authors here. In particular, explain why the coordinate functions x and y are in $\mathcal{A}_v$, why $x - a$ and $y - b$ are in M_v, etc, including the claims about the infinite valuation.

(5) In defining the Galois action on a Riemann surface, your authors write that the bijection $S \leftrightarrow S^\sigma$, defined by the Galois element σ, is not holomorphic.

(a) Prove that the map from S_F to S_{F^σ}, explicitly defined on page 196, is an isomorphism.

(b) Explain why the bijection defined by σ is not holomorphic.

(6) *Write* a few paragraphs in which you outline the main ideas from chapter 3, explaining how they are used to prove Belyi's Theorem.