MAT 401 HOMEWORK–3 DUE *IN CLASS*, AT THE BEGINNING OF CLASS, ON THURSDAY, OCTOBER 17

One goal for this course is for you to develop your skill in effectively communicating mathematics. With this in mind, you should clearly write up your solutions. Solutions with little or no justification will receive little or no credit.

(1) Julian explained how one can realize the torus as a quotient of $\mathbb{C}$, its universal cover, by the group $\omega_1\mathbb{Z} \times \omega_1\mathbb{Z}$, where ω_1 and ω_2 are linearly independent vectors in $\mathbb{C}$.

   (a) Describe the automorphism group of this covering map $\pi : \mathbb{C} \to \mathbb{C}/(\omega_1\mathbb{Z} \times \omega_1\mathbb{Z})$.

   (b) Your authors write, (page 15), that the automorphisms of $\mathbb{C}$ consist of the maps $z \mapsto az + b$. Are these automorphisms the same as the automorphism group $\mathrm{Aut}(\mathbb{C}, \pi)$, described above? Clearly explain.

   (c) Find 3 subgroups of $(\mathrm{Aut}(\mathbb{C}), \pi)$, and describe the corresponding intermediate covers of the torus, as in Theorem 1.69(v). What are the fundamental groups of the intermediate covers.

(2) On the first HW you proved that the map from $X : y^2 = x^8 - 1$ to $X' : y^2 = x^5 - x$ given by $(x, y) \mapsto (x^2, xy)$ is unramified, i.e., that it is a covering map.

   Carefully explain how to realize this covering map as a quotient $X \to X/G \cong X'$?

(3) Belyi's theorem states that a Riemann surface $\mathcal{S}$ can be defined by an equation with coefficients in $\overline{\mathbb{Q}}$ precisely when there exists a morphism $\mathcal{S} \to \mathbb{P}^1$ which is ramified over exactly 3 points, which we can assume to be $\{0, 1, \infty\}$. Prove that we can indeed map any 3 points of $\mathbb{P}^1$ onto $\{0, 1, \infty\}$ by an automorphism of $\mathbb{P}^1$.

(4) This problem refers to the first part of the proof of Belyi's theorem that we discussed in class. Using the authors' notation, explain why the points $m_1(\{\text{roots of } m_1'\})$ are branching values of $m_1 : \mathbb{P}^1 \to \mathbb{P}^1$.

(5) Consider the Riemann surfaces defined by the equations below:

$$
\begin{aligned}
y^2 &= x^3 \\
y^2 &= x^3 + 1 \\
y &= x^2 \\
y^2 &= x^2(x + 1) \\
y^2 &= x(x + 1)(x - 1)
\end{aligned}
$$

   For each surface, give a set of generators for its field of meromorphic functions. Show that some of these equations define isomorphic Riemann surfaces, by determining appropriate relations among the generators.