1. Introduction and Overview of Main Results

I am interested in quasiconformal geometry and its applications to complex dynamics, in particular the dynamics of transcendental functions. A transcendental function is a holomorphic map \( f : \mathbb{C} \to \mathbb{C} \) which is not a polynomial. The existence of wandering domains for various classes of holomorphic maps has been well studied in complex dynamics. The first example of a wandering domain for an entire function is due to Baker [Bak76], who constructed such an example using infinite products. Sullivan proved, on the other hand, that wandering domains do not occur for rational maps [Sul85]. This no-wandering domain theorem was extended to a subclass of transcendental functions we now will describe.

A singular value of a map \( f : \mathbb{C} \to \mathbb{C} \) is a critical value or an asymptotic value. An asymptotic value of \( f \) is some \( w \in \mathbb{C} \) so that there exists a curve \( \gamma : [0, \infty) \to \mathbb{C} \), with \( \gamma(t) \to \infty \) as \( t \to \infty \), so that \( f(\gamma(t)) \to w \) as \( t \to \infty \). For example 0 is an asymptotic value of \( \exp \) since this definition is satisfied for any curve whose real part tends to negative infinity. The singular values of \( f \), denoted \( S(f) \), are the places where one is unable to define a local inverse for \( f \). The class of transcendental functions with finitely many singular values is called the Speiser class, denoted \( S \). Two groups of mathematicians, Eremenko and Lyubich [EL84] and Goldberg and Keen [GK86], proved the no-wandering domain theorem for the class \( S \). Both no-wandering domain theorems were proven using techniques from quasiconformal geometry.

A more general class of transcendental functions, called the Eremenko-Lyubich class and denoted by \( \mathcal{B} \), consists of those functions \( f \) with bounded (but not necessarily finite) \( S(f) \). Whether wandering domains occur in \( \mathcal{B} \) was answered by Bishop [Bis15], who indeed constructed functions in \( \mathcal{B} \) with wandering domain. Much of the theory for the class \( \mathcal{B} \) was developed in [EL92]. There it was proven that in class \( \mathcal{B} \), Fatou components can not escape uniformly to infinity. In particular this implies that in order for a wandering domain to occur in class \( \mathcal{B} \) it would have to oscillate - i.e. return to some compact subset of the plane infinitely often. The first example of such an oscillating wandering domain was indeed given in [EL87], however this function was not in class \( \mathcal{B} \). [EL87] uses approximation theory and also contains several other constructions of transcendental functions with pathological dynamics. [Her84] also contains a relevant construction of a transcendental function with a simply connected wandering domain.

Bishop’s construction of a wandering domain in class \( \mathcal{B} \), on the other hand, relies on the folding theorem, proven in the same paper. My thesis [Laz17] investigates the topology of the wandering domain in Bishop’s example. Furthermore, my thesis uses Bishop’s folding techniques to answer questions about the existence of wandering domains for transcendental functions with certain specified properties. We now describe these questions more precisely.

The lambda-limit set \( \Lambda(z, f) \) for a point \( z \in \mathbb{C} \), and a function \( f \) is the accumulation set of the sequence \( (f^n(z))_{n \geq 0} \). If \( z, w \) both lie in the same Fatou component \( U \) of the function \( f \), it is a theorem of Fatou [Fat20] that \( \Lambda(z, f) = \Lambda(w, f) \). So in fact we may write \( \Lambda(U, f) \) unambiguously. Osborne and Sixsmith [OS16] asked the following:
Is there a transcendental entire function \( f \) with a wandering domain \( U \) so that \( \Lambda(U, f) \) is uncountable?

In Bishop’s example, \( \Lambda(U, f) \) consists of the point \( 1/2 \) and its forward orbit (so that \( \Lambda(U, f) \) is countable in this case). My thesis modifies the construction to yield a function \( f \) where \( \Lambda(U, f) \) consists of the interval \([1/2, 5/8]\) and its forward orbit. This answers the above question in the affirmative. A natural question about Bishop’s example (or the above modification) is whether the Fatou components that comprise the wandering domain are bounded or not. My thesis proves that these Fatou components are bounded (although their orbit is unbounded). In view of this there is another question asked in the same paper of Osborne and Sixsmith [OS16]:

Is there a transcendental entire function with an unbounded wandering domain in \( BU(f) \), all of whose iterates are unbounded?

Here \( BU(f) \) is the set of \( z \in \mathbb{C} \) whose orbits are neither escaping to infinity nor staying in a bounded subregion of the plane. Indeed [OS16] studies the dynamical partition of \( \mathbb{C} \) into \( BU(f) \) and its complement. Using Bishop’s folding techniques, my thesis constructs a function in \( \mathcal{B} \) with wandering domain, so that each Fatou component comprising the wandering domain is unbounded. This construction, together with Theorem 1.1 from [OS16], gives an affirmative answer to the above question.

2. The dynamics of wandering domains in \( \mathcal{B} \)

We will now describe briefly the dynamics of Bishop’s wandering domain in \( \mathcal{B} \). We build the function \( f \). We start with the region \( S^+ \) pictured in Figure 1. We define \( f \) in \( S^+ \) by mapping conformally to the right half plane \( \mathbb{H}_r \) by \( \lambda \sinh \) for any \( \lambda \in \pi \mathbb{N}^+ \). We follow this by \( \cosh \) which maps \( \mathbb{H}_r \) holomorphically onto \( \mathbb{C} \setminus [-1, 1] \).

![Figure 1](image-url)
Consider a sequence of discs \((D_n)_{n \geq 1}\) of radius 1 placed above the region \(S^+\) as pictured in Figure 2. We will also keep track of the orbit of \(1/2\) under \(f\). Let \(D_{p_n}\) be the disc closest to \(f^n(1/2)\). Notice that there is a preimage of \(D_{p_3}\) under \(z \rightarrow \cosh(\lambda \sinh(z))\) close to \(f^2(1/2)\). By taking further preimages we see that there is a (third) preimage of the disc \(D_{p_3}\) near \(1/2\), as pictured in Figure 2.

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{figure2.png}
\end{array}
\]

**Figure 2**

We will define our function \(f\) in \(D_{p_2}\) so that for a subdisc \(\tilde{D}_{p_2} \subset D_{p_2}\), we have \(f(\tilde{D}_{p_2}) \subset f^{-3}(\tilde{D}_{p_3})\). These subdiscs will eventually be part of a wandering domain. For \(z \in D_n\), we define \(f(z) = \rho_n((z - z_n)^{d_n})\). This map is illustrated in Figure 3. \(z_n\) is the center of \(D_n\), \(d_n\) is a parameter that will allow us to control how small the image of \(\tilde{D}_n\) will be, and \(\rho_n\) is a quasiconformal map perturbing the origin that is the identity on \(\partial \mathbb{D}\) and is conformal on \((3/4)\mathbb{D}\).

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{figure3.png}
\end{array}
\]

**Figure 3**

The resulting map is illustrated in Figure 4. We perform this procedure over the entire sequence of discs \((D_{p_n})\) so that the subdisc \(\tilde{D}_{p_n} \subset D_{p_n}\) is mapped by \(f\) into a \((n + 1)\)st preimage of \(\tilde{D}_{p_{n+1}}\) near \(1/2\). This means our wandering domain will follow the orbit of \(1/2\) for \((n - 1)\) iterations inside \(S^+\) before landing (at the \(n^{\text{th}}\) iterate) in a disc \(\tilde{D}_{p_n}\) whence it is mapped back near \(1/2\). Then the
wandering domain follows the orbit of $1/2$ for $n$ iterations inside $S^+$ before landing (at the $(n+1)^{st}$ iterate) in $D_{p_{n+1}}$, whence it is mapped back near $1/2$, and so on...

$$z \rightarrow \sinh(\lambda \cosh(z))$$

**Figure 4**

Bishop’s folding theorem provides the machinery with which we can produce an entire function that mimics the dynamics of the above $f$. Above, we have described a definition of $f$ on $S^+$ and the discs $(D_n)$. The folding theorem extends this definition to all of $\mathbb{C}$ so that $f$ is quasiregular. The measurable Riemann-mapping theorem is then invoked to produce a quasiconformal $\phi$ so that $f \circ \phi$ is holomorphic. Moreover the quasiconformality constant of $\phi$ is independent of the parameters $(\lambda, (d_n))$. And in fact as the parameters $(\lambda, (d_n))$ increase, the support of the beltrami coefficient of $\phi$ shrinks in area to zero. This means we can ensure $\phi$ is close enough to the identity so that the wandering domain described above for the quasiregular function $f$ will still occur for the entire function $f \circ \phi$.

We finish this section by describing the author’s own contributions in greater detail. The above construction will be modified to yield a transcendental $f$ with wandering domain $U$ so that $\Lambda(U,f)$ is uncountable (answering a question of [OS16]). In the previous construction a disc followed the iteration of $1/2$ until it landed in $D_{p_n}$, whence we sent it back to a preimage of $D_{p_{n+1}}$. Recall $D_{p_{n+1}}$ was the closest disc to the $(n+1)^{st}$ iterate of $1/2$. Instead of sending $D_{p_n}$ back to a $(n+1)^{st}$ preimage of a disc near $f^{n+1}(1/2)$, we will send it to a $(n+1)^{st}$ preimage of a disc near $f^{n+1}(5/8)$. This means after following the orbit of $1/2$ for $n$ iterates, our disc will follow the orbit of $5/8$ for $n+1$ iterates. After this we send it back to a $(n+2)^{nd}$ preimage of some disc near $f^{n+2}(x_{n+2})$ where $x_{n+2} \in [1/2, 5/8]$. Performing this procedure over some dense subset $(x_n) \in [1/2, 5/8]$ yields an $f$ with uncountable $\Lambda(U,f)$. Indeed, the accumulation set of our wandering domain will be $[1/2, 5/8]$ and its forward iteration under $f$. This procedure is illustrated in Figure 5.
Again the folding theorem is used to extend the above definition of $f$ to all of $\mathbb{C}$ so that $f$ is quasiregular. And again the measurable Riemann mapping theorem is invoked to produce quasiconformal $\phi$ so that $f \circ \phi$ is entire. As in Bishop’s example one must argue that the parameters may be chosen so that $\phi$ is close enough to the identity in order for $f \circ \phi$ to still have the above described wandering domain.

One may ask for a better description of the wandering domain we described above. We argued that these subdiscs ($\tilde{D}_p$) belonged to a wandering domain, but a priori it could be the case the Fatou component containing ($\tilde{D}_p$) is much larger. In particular could the Fatou component containing any ($\tilde{D}_p$) be unbounded? In fact the answer is no, neither the wandering domain in Bishop’s original example nor the wandering domain in the above modification can have unbounded Fatou components. The proof is contained in my thesis and involves basic tools in hyperbolic geometry and a deeper understanding of the way $f$ is extended quasiregularly to $\mathbb{C}$.

Lastly in this section we describe a construction in my thesis similar to those given above that yields a wandering domain whose Fatou components are unbounded (answering a question of [OS16]). We keep the same definition of $f$ in $S^+$ given in the previous sections, but replace the discs ($D_n$) with strips ($S_n$) (containing the discs ($D_n$)) as illustrated in Figure 6. As before we consider a preimage of $D_{p_3}$ near $1/2$.

Notice that we can map each $S_n$ by a Euclidean similarity to $S^+$, whence it is mapped conformally to the left half-plane by $z \rightarrow -\lambda_n \sinh(z)$ where $\lambda_n > 0$. We follow this by by the exponential and then a quasiconformal map $\rho_n$ described earlier in this section. This procedure is illustrated in Figure 7. In this construction the parameters ($\lambda_n$) play the role of ($d_n$) in the earlier constructions.
- namely they allow us to shrink the size of the image of some unbounded substrip \( \tilde{S}_n \subset S_n \). The domains \((\tilde{S}_{p_n})\) are proven to belong to a wandering domain of \( f \).

We choose the parameter \( \lambda_{p_2} \) large enough so that the diameter of \( f(\tilde{S}_{p_2}) \) is much smaller than the diameter of \( f^{-3}(D_{p_3}) \). Then \( \rho_{p_2} \) is chosen as before so that \( f(\tilde{S}_{p_2}) \subset f^{-3}(D_{p_3}) \). This is illustrated in Figure 8. We perform this procedure over the entire sequence \((\tilde{S}_{p_n}), (D_{p_n})\). Then we apply once again Bishop’s folding theorem to extend \( f \) to be quasiregular on \( \mathbb{C} \) and finally apply the measurable Riemann mapping theorem to produce quasiconformal \( \phi \) so that \( f \circ \phi \) is entire. If the above parameters are chosen carefully, one can prove that \( f \circ \phi \) still has the above described wandering domain.
3. Further work

One promising problem I am working on (using similar methods to those described above) is to construct a function in class $B$ with wandering domain and finite order. In the above examples our maps were roughly of the form $z \to \exp(\exp(z))$ in $S^+$ which is of infinite order. The general strategy is in defining $f(z) = \exp(z)$ in the right half plane, and defining a sequence of discs ($D_n$) along the negative real axis so that $f(D_n) \subset f^{-n+1}(D_{n+1})$, as pictured in Figure 9. Again we use maps of the form $z \to (z - z_n)^{d_n}$ to shrink subdiscs of $D_n$ in size. The difficulty is in that if we impose finite order on our function we lose the freedom to increase arbitrarily the size of the exponents ($d_n$) as in the previously described constructions. Nevertheless we can extend the above $f$ to be quasiregular on $\mathbb{C}$ (using Bishop’s theorem), so that $f$ has wandering domain. However the inability to increase the size of the exponents ($d_n$) results in less control over the correction map $\phi$ when applying the measurable Riemann mapping theorem. So it is difficult to prove that although $f$ has a wandering domain, this property still holds for $f \circ \phi$. 

**Figure 8**
Figure 9
References


