MAT 552: Lie groups and Lie algebras
Fall 2004

Lecture 1: (8/31) Introduction. Definition of a Lie group; C^1 implies analytic. Examples: $\mathbb{R}^n, S^1, \mathrm{SU}(2)$. Theorem about closed subgroup (no proof). Connected component and universal cover. G/H.

Lecture 2: (9/2) Action of G on manifolds; homogeneous spaces. Action on functions, vector fields, etc. Left, right, and adjoint action. Representations.

Lecture 3: (9/7) Classical groups: GL, SL, SU, SO, Sp – definition. Exponential and logarithmic mapping for matrix groups. Proof that classical groups are smooth; calculation of the corresp. Lie algebra and dimension. Topological information (connectedness, π_1).

Lecture 4: (9/9) Lie algebra of a Lie groups: $\mathfrak{g} = T_eG$ = right-invariant vector fields = 1-parameter subgroups. Exponential and logarithmic maps. Morphisms $f\colon G_1 \to G_2$ are determined by $f_*\colon \mathfrak{g}_1 \to \mathfrak{g}_2$. Adjoint action of G on $\mathfrak{g}$. Example: elements $J_x, J_y, J_z \in \mathfrak{so}(3)$.

Lecture 5: (9/14) Commutator, $e^x e^y = e^{x+y+\frac{1}{2}[x,y]+\cdots}$. Relation with group commutator and commutator of vector fields. $[x, y] = xy - yx$ for matrix groups. Example: $\mathfrak{so}(3)$. Jacobi identity. Abstract Lie algebras and morphisms. Campbell–Hausdorff formula (without proof).

Lecture 6: (9/21) $\mathrm{Hom}(G_1, G_2) \hookrightarrow \mathrm{Hom}(\mathfrak{g}_1, \mathfrak{g}_2)$. If G_1 is simply-connected, then $\mathrm{Hom}(G_1, G_2) = \mathrm{Hom}(\mathfrak{g}_1, \mathfrak{g}_2)$. Analytic subgroups and Lie subalgebras. Ideals in $\mathfrak{g}$ and normal subgroups in G.

Lecture 7: (9/23) Lie's third theorem (no proof). Corollary: category of connected, s.c. Lie groups is equivalent to the category of Lie algebras. Representations of G = representations of $\mathfrak{g}$. Action by vector fields. Example: representations of $\mathrm{SO}(3), \mathrm{SU}(2)$. Complexification; $\mathfrak{su}(n)$ and $\mathfrak{sl}(n)$.

Lecture 8: (9/28) Universal enveloping algebra. Poincare-Birkhoff-Witt theorem. Casimir element in $U\mathfrak{sl}(2)$.

Lecture 9: (9/30) Group and algebra representations. Subrepresentations, direct sums, $V_1 \otimes V_2$, V^*, action on $\mathrm{End}\, V$. Irreducibility. Intertwining operators. Schur lemma. Semisimplicity.

Lecture 10: (10/5) Unitary representations. Complete reducibility of representation for a group with invariant integral. Invariant integral for finite group and for compact Lie groups; Haar measure.

Lecture 11: (10/7) Examples: representations of $\mathbb{Z}_n$, S_3, $\mathbb{R}$ and S^1; Fourier series as decomposition of a representation into irreducibles.

Lecture 12: (10/12) Characters and Peter–Weyl theorem.

Lecture 13: (10/14) Solvable and nilpotent Lie algebras: equivalent definitions. Example: upper triangular matrices.

Lecture 14: (10/19) Lie theorem (about representations of a solvable Lie algebra). Engel's theorem (without proof). Commutant and radical. Semisimple Lie algebras. Levi theorem (without proof).

Lecture 15: (10/21) Invariant bilinear forms. Example: trace in a representation. Cartan's criterion of solvability (without proof) and semisimplicity. Example: semisimplicity of $\mathfrak{sl}(2)$.

Lecture 16: (10/26) Semisimple Lie algebra is a direct sum of simple. Reductive Lie algebras. Reductivity of Lie algebra of a compact Lie group. Semisimplicity of classical Lie algebras.

Lecture 17: (10/28) Casimir element and complete reducibility of representations of a semisimple Lie algebra.

Lecture 18: (11/2) Representations of $\mathfrak{sl}(2)$.

Lecture 19: (11/4) Semisimple and nilpotent elements; Jordan decomposition. Toral subalgebras. Definition of Cartan (a.k.a. maximal toral) subalgebra. Theorem: conjugacy of Cartan subalgebras (no proof).

Lecture 20: (11/9) Root decomposition and root system for semisimple Lie algebra. Basic properties. Example: $\mathfrak{sl}(n)$.

Lecture 21: (11/11) Properties of root system for s.s. Lie algebra continued. Definition of an abstract root system. Irreducible root systems and simple Lie algebras. Example: $\mathfrak{so}(4)$.

Lecture 22: (11/16) Classification of rank 2 root systems. Simple roots and their properties. Cartan matrix.

Lecture 23: (11/18) Dynkin diagrams. Classification of Dynkin diagrams (partial proof). Definition of Weyl group.

Lecture 24: (11/23) Simple reflections and Weyl group. Weyl chambers. Reconstructing root system from set of simple roots. Transitivity of action of W on the set of Weyl chambers. Length $l(w)$ and its geoemtric interpretation as number of separating hyperplanes.

Lecture 25: (11/30) Constructing a semisimple Lie algebra from a root system. Serre relations and Serre theorem (no proof). Classification of simple Lie algebras.

Lecture 26: (12/2) Finite-dimensional representations of a semi-simple Lie algebra. Weights; symmetry under Weyl group. Example: $\mathfrak{sl}(3)$. Singular vectors.

Lecture 27: (12/7) Verma modules and irreducible highest weight modules. Dominant weights and classification of finite-dimensional highest weight modules (without proof)

Lecture 28: (12/9) Example: representations of $\mathfrak{sl}(n)$.