Lecture 1: (8/31) Introduction. Definition of a Lie group; $C^1$ implies analytic. Examples: $\mathbb{R}^n, S^1, SU(2)$. Theorem about closed subgroup (no proof). Connected component and universal cover. $G/H$.

Lecture 2: (9/2) Action of $G$ on manifolds; homogeneous spaces. Action on functions, vector fields, etc. Left, right, and adjoint action. Representations.

Lecture 3: (9/7) Classical groups: $GL, SL, SU, SO, Sp$ – definition. Exponential and logarithmic mapping for matrix groups. Proof that classical groups are smooth; calculation of the corresponding Lie algebra and dimension. Topological information (connectedness, $\pi_1$).

Lecture 4: (9/9) Lie algebra of a Lie group: $\mathfrak{g} = T_e G = $ right-invariant vector fields = 1-parameter subgroups. Exponential and logarithmic maps. Morphisms $f: G_1 \to G_2$ are determined by $f_*: \mathfrak{g}_1 \to \mathfrak{g}_2$. Adjoint action of $G$ on $\mathfrak{g}$. Example: elements $J_x, J_y, J_z \in \mathfrak{so}(3)$.

Lecture 5: (9/14) Commutator, $e^x e^y = e^{x+y+\frac{1}{2}[x,y]+\cdots}$. Relation with group commutator and commutator of vector fields. $[x,y] = xy - yx$ for matrix groups. Example: $\mathfrak{so}(3)$. Jacobi identity. Abstract Lie algebras and morphisms. Campbell–Hausdorff formula (without proof).

Lecture 6: (9/21) $\text{Hom}(G_1, G_2) \hookrightarrow \text{Hom}(\mathfrak{g}_1, \mathfrak{g}_2)$. If $G_1$ is simply-connected, then $\text{Hom}(G_1, G_2) = \text{Hom}(\mathfrak{g}_1, \mathfrak{g}_2)$. Analytic subgroups and Lie subalgebras. Ideals in $\mathfrak{g}$ and normal subgroups in $G$.

Lecture 7: (9/23) Lie’s third theorem (no proof). Corollary: category of connected, s.c. Lie groups is equivalent to the category of Lie algebras. Representations of $G = $ representations of $\mathfrak{g}$. Action by vector fields. Example: representations of $SO(3), SU(2)$. Complexification; $\mathfrak{su}(n)$ and $\mathfrak{sl}(n)$.

Lecture 8: (9/28) Universal enveloping algebra. Poincare-Birkhoff-Witt theorem. Casimir element in $U\mathfrak{sl}(2)$.

Lecture 9: (9/30) Group and algebra representations. Subrepresentations, direct sums, $V_1 \otimes V_2$, $V^*$, action on $\text{End} V$. Irreducibility. Intertwining operators. Schur lemma. Semisimplicity.

Lecture 10: (10/5) Unitary representations. Complete reducibility of representation for a group with invariant integral. Invariant integral for finite group and for compact Lie groups; Haar measure.

Lecture 11: (10/7) Examples: representations of $\mathbb{Z}_n$, $S_3$, $\mathbb{R}$ and $S^1$; Fourier series as decomposition of a representation into irreducibles.

Lecture 12: (10/12) Characters and Peter–Weyl theorem.


Lecture 15: (10/21) Invariant bilinear forms. Example: trace in a representation. Cartan’s criterion of solvability (without proof) and semisimplicity. Example: semisimplicity of $\mathfrak{sl}(2)$.


Lecture 17: (10/28) Casimir element and complete reducibility of representations of a semisimple Lie algebra.

Lecture 18: (11/2) Representations of $\mathfrak{sl}(2)$.


Lecture 20: (11/9) Root decomposition and root system for semisimple Lie algebra. Basic properties. Example: $\mathfrak{sl}(n)$.


Lecture 26: (12/2) Finite-dimensional representations of a semi-simple Lie algebra. Weights; symmetry under Weyl group. Example: $\mathfrak{sl}(3)$. Singular vectors.

Lecture 27: (12/7) Verma modules and irreducible highest weight modules. Dominant weights and classification of finite-dimensional highest weight modules (without proof).

Lecture 28: (12/9) Example: representations of $\mathfrak{sl}(n)$.