MAT 552: Lie groups and Lie algebras
Fall 2004

Lecture 1: (8/31) Introduction. Definition of a Lie group; \( C^1 \) implies analytic. Examples: \( \mathbb{R}^n, S^1, \text{SU}(2) \). Theorem about closed subgroup (no proof). Connected component and universal cover.

Lecture 2: (9/2) \( G/H \). Action of \( G \) on manifolds; homogeneous spaces. Action on functions, vector fields, etc. Left, right, and adjoint action. Left, right, and bi-invariant vector fields (forms, etc).

Lecture 3: (9/7) Classical groups: \( GL, SL, SU, SO, Sp \) – definition. Exponential and logarithmic mapping for matrix groups. Proof that classical groups are smooth; calculation of the corresp. Lie algebra and dimension. Topological information (connectedness, \( \pi_1 \)). One-parameter subgroups in a Lie group: existence and uniqueness.

Lecture 4: (9/9) Lie algebra of a Lie groups: \( \mathfrak{g} = T_eG \) = right-invariant vector fields = 1-parameter subgroups. Exponential and logarithmic maps and their properties. Morphisms \( f: G_1 \to G_2 \) are determined by \( f_*: \mathfrak{g}_1 \to \mathfrak{g}_2 \). Example: elements \( J_x, J_y, J_z \in \mathfrak{so}(3) \). Definition of commutator: \( e^x e^y = e^{x+y+\frac{1}{2}[x,y]+...} \).

Lecture 5: (9/14) Properties of commutator. Relation with group commutator; \( \text{Ad} \) and \( \text{ad} \). Jacobi identity. Abstract Lie algebras and morphisms. \([x,y] = xy - yx \) for matrix groups. Relation with commutator of vector fields. Campbell–Hausdorff formula (without proof).

Lecture 6: (9/21) If \( G_1 \) is simply-connected, then \( \text{Hom}(G_1, G_2) = \text{Hom}(\mathfrak{g}_1, \mathfrak{g}_2) \). Immersed subgroups and Lie subalgebras. Ideals in \( \mathfrak{g} \) and normal subgroups in \( G \).

Lecture 7: (9/23) Lie’s third theorem (no proof). Corollary: category of connected, s.c. Lie groups is equivalent to the category of Lie algebras. Representations of \( G = \) representations of \( \mathfrak{g} \). Action by vector fields. Example: representations of \( \text{SO}(3), \text{SU}(2) \). Complexification; \( \mathfrak{su}(n) \) and \( \mathfrak{sl}(n) \).

Lecture 8: (9/28) Representations of Lie groups and Lie algebras. Subrepresentations, direct sums, \( V_1 \otimes V_2 \), \( V^* \), action on \( \text{End} V \). Irreducibility. Intertwining operators. Schur lemma. Semisimplicity.

Lecture 9: (9/30) Unitary representations. Complete reducibility of representation for a group with invariant integral. Invariant integral for finite group and for compact Lie groups; Haar measure. Example: representations of \( S^1 \) and Fourier series.

Lecture 10: (10/5) Characters and Peter–Weyl theorem.

Lecture 11: (10/7) Universal enveloping algebra. Central element \( J_x^2 + J_y^2 + J_z^2 \in U \mathfrak{so}(3, \mathbb{R}) \). Statement of PBW theorem.


Lecture 14: (10/19) Jacobi decomposition (into semisimple and nilpotent element). Proof of Cartan criterion.
Lecture 15: (10/21) Corollaries: every semisimple algebra is direct sum of simple ones; ideal, quotient of a s.s. is semisimple; $[g, g] = g$; every derivation is inner. Relation between s.s. Lie algebras and compact groups.

Lecture 16: (10/26) Complete reducibility of representations

Lecture 17: (10/28) Representations of $\mathfrak{sl}(2)$. Semisimple elements in a Lie algebra.


Lecture 19: (11/4) Root decomposition and root system for semisimple Lie algebra. Basic properties. Example: $\mathfrak{sl}(n)$.
