MAT 552: Lie groups and Lie algebras
Fall 2004

Lecture 1: (8/31) Introduction. Definition of a Lie group; C^1 implies analytic. Examples: $\mathbb{R}^n, S^1, \mathrm{SU}(2)$. Theorem about closed subgroup (no proof). Connected component and universal cover.

Lecture 2: (9/2) G/H. Action of G on manifolds; homogeneous spaces. Action on functions, vector fields, etc. Left, right, and adjoint action. Left, right, and bi-invariant vector fields (forms, etc).

Lecture 3: (9/7) Classical groups: GL, SL, SU, SO, Sp – definition. Exponential and logarithmic mapping for matrix groups. Proof that classical groups are smooth; calculation of the corresp. Lie algebra and dimension. Topological information (connectedness, π_1). One-parameter subgroups in a Lie group: existence and uniqueness.

Lecture 4: (9/9) Lie algebra of a Lie groups: $\mathfrak{g} = T_eG$ = right-invariant vector fields = 1-parameter subgroups. Exponential and logarithmic maps and their properties. Morphisms $f\colon G_1 \to G_2$ are determined by $f_*\colon \mathfrak{g}_1 \to \mathfrak{g}_2$. Example: elements $J_x, J_y, J_z \in \mathfrak{so}(3)$. Definition of commutator: $e^x e^y = e^{x+y+\frac{1}{2}[x,y]+\cdots}$.

Lecture 5: (9/14) Properties of commutator. Relation with group commutator; Ad and ad. Jacobi identity. Abstract Lie algebras and morphisms. $[x, y] = xy - yx$ for matrix groups. Relation with commutator of vector fields. Campbell–Hausdorff formula (without proof).

Lecture 6: (9/21) If G_1 is simply-connected, then $\mathrm{Hom}(G_1, G_2) = \mathrm{Hom}(\mathfrak{g}_1, \mathfrak{g}_2)$. Immersed subgroups and Lie subalgebras. Ideals in $\mathfrak{g}$ and normal subgroups in G.

Lecture 7: (9/23) Lie's third theorem (no proof). Corollary: category of connected, s.c. Lie groups is equivalent to the category of Lie algebras. Representations of G = representations of $\mathfrak{g}$. Action by vector fields. Example: representations of $\mathrm{SO}(3), \mathrm{SU}(2)$. Complexification; $\mathfrak{su}(n)$ and $\mathfrak{sl}(n)$.

Lecture 8: (9/28) Representations of Lie groups and Lie algebras. Subrepresentations, direct sums, $V_1 \otimes V_2$, V^*, action on $\mathrm{End}\, V$. Irreducibility. Intertwining operators. Schur lemma. Semisimplicity.

Lecture 9: (9/30) Unitary representations. Complete reducibility of representation for a group with invariant integral. Invariant integral for finite group and for compact Lie groups; Haar measure. Example: representations of S^1 and Fourier series.

Lecture 10: (10/5) Characters and Peter–Weyl theorem.

Lecture 11: (10/7) Universal enveloping algebra. Central element $J_x^2 + J_y^2 + J_z^2 \in U\ \mathfrak{so}(3, \mathbb{R})$. Statement of PBW theorem.

Lecture 12: (10/12) Structure theory of Lie algebras: generalitites. Commutant. Solvable and nilpotent Lie algebras: equivalent definitions. Example: upper triangular matrices. Lie theorem (about representations of a solvable Lie algebra).

Lecture 13: (10/14) Engel's theorem (without proof). Radical. Semisimple Lie algebras. Example: semisimplicity of $\mathfrak{sl}(2)$. Levi theorem (without proof). Statement of Cartan's criterion of solvability and semisimplicity.

Lecture 14: (10/19) Jacobi decomposition (into semisimple and nilpotent element). Proof of Cartan criterion.

Lecture 15: (10/21) Corollaries: every semisimple algbera is direct sum of simple ones; ideal, quotient of a s.s. is semisimple; $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$; every derivation is inner. Relation between s.s. Lie algberas and compact groups.

Lecture 16: (10/26) Complete reducibility of representations

Lecture 17: (10/28) Representations of $\mathfrak{sl}(2)$. Semisimple elements in a Lie algebra.

Lecture 18: (11/2) Semisimple and nilpotent elements; Jordan decomposition. Toral subalgebras. Definition of Cartan (a.k.a. maximal toral) subalgebra. Theorem: conjugacy of Cartan subalgebras (no proof).

Lecture 19: (11/4) Root decomposition and root system for semisimple Lie algebra. Basic properties. Example: $\mathfrak{sl}(n)$.

Lecture 20: (11/9) Definition of an abstract root system. Classification of rank 2 root systems. Positive roots and simple roots.

Lecture 21: (11/11) Weyl chambers and Weyl group. Transitivity of action of W on the set of Weyl chambers. Reconstructing root system from set of simple roots.