1. (a) Prove that $\mathbb{R}^3$, considered as Lie algebra with the commutator given by the cross-product, is isomorphic (as a Lie algebra) to $\mathfrak{so}(3, \mathbb{R})$.
(b) Let $\varphi: \mathfrak{so}(3, \mathbb{R}) \to \mathbb{R}^3$ be the isomorphism of part (a). Prove that under this isomorphism, the standard action of $\mathfrak{so}(3)$ on $\mathbb{R}^3$ is identified with the action of $\mathbb{R}^3$ on itself given by the cross-product:

$$a \cdot \vec{v} = \varphi(a) \times \vec{v}, \quad a \in \mathfrak{so}(3), \vec{v} \in \mathbb{R}^3$$

where $a \cdot \vec{v}$ is the usual multiplication of a matrix by a vector.

2. Write the commutation relations for the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ in the basis

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

3. Write explicitly Lie algebra isomorphisms $\mathfrak{su}(2) \simeq \mathfrak{so}(3, \mathbb{R})$, $\mathfrak{so}(3, \mathbb{R})|_\mathbb{C} \simeq \mathfrak{so}(3, \mathbb{C}) \simeq \mathfrak{sl}(2, \mathbb{C})$.

4. Let $\varphi: SU(2) \to SO(3, \mathbb{R})$ be the cover map constructed in Problem Set 1.
(a) Show that $\ker \varphi = \{1, -1\} = \{1, e^{\pi i \theta}\}$, where $\theta$ is defined in Problem 2.
(b) Using this, show that representations of $SO(3, \mathbb{R})$ are the same as representations of $\mathfrak{sl}(2, \mathbb{C})$ satisfying $e^{\pi i \theta} = 1$.

5. Let $\mathcal{C}_n$ be the space of polynomials with real coefficients of degree $\leq n$ in variable $x$. The Lie group $G = \mathbb{R}$ acts on $\mathcal{C}_n$ by translations of the argument: $\rho(t)(x) = x + t, t \in G$. Show that the corresponding action of the Lie algebra $\mathfrak{g} = \mathbb{R}$ is given by $\rho(a) = a\partial_x, a \in \mathfrak{g}$ and deduce from this the Taylor formula for polynomials:

$$f(x + t) = \sum_{n \geq 0} \frac{(t\partial_x)^n}{n!} f$$

6. Let $\text{SL}(2, \mathbb{C})$ act on $\mathbb{P}^1$ in the usual way:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} (x : y) = (ax + by : cx + dy)$$

This defines an action of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ by vector fields on $\mathbb{P}^1$. Write explicitly vector fields corresponding to $h, e, f$ in terms of coordinate $t = x/y$ on the open cell $\mathbb{C} \subset \mathbb{P}^1$.

7. Let $J_x, J_y, J_z$ be the basis in $\mathfrak{so}(3, \mathbb{R})$ described in class. The standard action of $SO(3, \mathbb{R})$ on $\mathbb{R}^3$ defines an action of $\mathfrak{so}(3, \mathbb{R})$ by vector fields on $\mathbb{R}^3$. Abusing the language, we will use the same notation $J_x, J_y, J_z$ for the corresponding vector fields on $\mathbb{R}^3$. Let $\Delta_{\text{sph}} = J_x^2 + J_y^2 + J_z^2$; this is a second order differential operator on $\mathbb{R}^3$, which is usually called the spherical Laplace operator, or the Laplace operator on the sphere.
(a) Write $\Delta_{\text{sph}}$ in terms of $x, y, z, \partial_x, \partial_y, \partial_z$.
(b) Show that $\Delta_{\text{sph}}$ is well defined as a differential operator on a sphere $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$, i.e., if $f$ is a function on $\mathbb{R}^3$ then $(\Delta_{\text{sph}} f)|_{S^2}$ only depends on $f|_{S^2}$.
(c) Show that $\Delta_{\text{sph}}$ is rotation invariant: for any function $f$ and $g \in SO(3, \mathbb{R})$, $\Delta_{\text{sph}}(gf) = g(\Delta_{\text{sph}} f)$.
*d) Show that the usual Laplace operator $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ can be written in the form $\Delta = \frac{1}{r^2} \Delta_{\text{sph}} + \Delta_{\text{radial}}$, where $\Delta_{\text{radial}}$ is a differential operator written in terms of $r = \sqrt{x^2 + y^2 + z^2}$ and $\partial_r$. 