MAT 534: HOMEWORK 9

DUE WED, NOV 21

Throughout this assignment, $\mathbb{F}$ is an arbitrary field.

1. Let $I \subset R$ be an ideal; let $\pi \colon R \to \overline{R} = R/I$ be the canonical homomorphism.
  (a) Show that the formula $J = \pi^{-1}(\overline{J})$ defines a bijection between ideals $J \supset I$ in R and ideals $\overline{J} \subset \overline{R}$.
  (b) Show that if $J = \pi^{-1}(\overline{J})$ are as in the previous problem, then $R/J \simeq \overline{R}/\overline{J}$.

2. Let $p \in \mathbb{R}[x]$ be a quadratic polynomial which has no real roots. Define $R = \mathbb{R}[x]/(p)$.
  (a) Show that $R \simeq \mathbb{C}$.
  (b) Show that $R \simeq \mathbb{R}[x, x^{-1}]/(p)$

3. Let $I = (x - y)$, $J = (x + y)$ be ideals in $\mathbb{C}[x, y]$.
  (a) Describe explicitly the rings $\mathbb{C}[x, y]/I$, $\mathbb{C}[x, y]/J$, $\mathbb{C}[x, y]/I + J$, $\mathbb{C}[x, y]/IJ$. (Hint: you may make change of variables $x' = x + y, y' = x - y$). Describe each of these rings as polynomial functions on a certain subset in $\mathbb{C}^2$.
  (b) Which of the ideals $I, J, I + J, IJ$ is maximal? prime?

4. From the textbook: problem 17 on p. 258

5. (a) From the textbook: problem 37 on p. 258.
  (b) Show that the ring $\mathbb{F}[[x]]$ of formal power series is local. Describe the maximal ideal.
  (c) Let $R \subset \mathbb{Q}$ be the set of all fractions which can be written in the form p/q, $p, q \in \mathbb{Z}$, q is odd. Prove that it is a local ring, with maximal ideal (2).

6. Let $a, b \in \mathbb{F}$, $a \neq b$. Prove that then ideals $(x - a)$ and $(x - b)$ in $\mathbb{F}[x]$ are comaximal. Deduce from this and Chinese remainder theorem that for any collection $a_1, \ldots, a_n \in \mathbb{F}$ with $a_i \neq a_j$ and any collection $c_1, \ldots, c_n \in \mathbb{F}$, there exists a polynomial $p \in \mathbb{F}[x]$ such that $p(a_i) = c_i$, and that such a polynomial is unique up to adding a multiple of $(x - a_1) \ldots (x - a_n)$.