Throughout this assignment, \( \mathbb{F} \) is an arbitrary field.

1. Which of the following rings are fields? integral domains? In each case, find all invertible elements (also called \textit{units})
   
   (a) \( R = \mathbb{F}[x] \)
   
   (b) \( R = \mathbb{Z}[\omega] \), where \( \omega \in \mathbb{C} \) is a primitive cubic root of unity.
   
   (c) \( R = \mathbb{R}[A] \) where \( A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \)
   
   (d) \( R = \mathbb{R}[A] \) where \( A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \)
   
   (e) \( R = \mathbb{Z}/n\mathbb{Z} \)

2. For each of the quotient rings below, answer the following questions: is it a field? is it finite? is it isomorphic to any ring of Problem 1?
   
   (a) \( \mathbb{Z}[i]/(2) \) 
   
   (b) \( \mathbb{Z}[i]/(i+1) \) 
   
   (c) \( \mathbb{R}[x]/(x-1)^2 \) 
   
   (d) \( \mathbb{R}[x]/(x^2+1) \) 
   
   (e) \( \mathbb{Z}[x]/(2,x) \) 
   
   (f) \( \mathbb{R}[x,y]/(xy) \)

3. Let \( d \in \mathbb{Z}, d > 1 \) be squarefree (i.e., \( d \) is not divisible by a square of any prime number).
   
   (a) Show that \( \mathbb{Q}[^{\sqrt{d}}] = \{ a + b\sqrt{d}, a, b \in \mathbb{Q} \} \) is a field.
   
   (b) Show that \( \mathbb{Z}[^{\sqrt{d}}] = \{ a + b\sqrt{d}, a, b \in \mathbb{Q} \} \) is an integral domain.
   
   (c) Define “conjugation” \( \overline{x+y} = x+y \in \mathbb{Q}[^{\sqrt{d}}] \) by \( \overline{a + b\sqrt{d}} = a - b\sqrt{d} \). Prove that \( \overline{x+y} = \overline{x} + \overline{y} \).
   
   (d) Show that \( u \in \mathbb{Z}[\sqrt{d}] \) is a unit (i.e., has a multiplicative inverse in \( \mathbb{Z}[\sqrt{d}] \)) iff \( u\overline{u} = \pm 1 \).

4. Using the previous problem, show that the set of all solutions of the \textit{Pell equation} \( a^2 - db^2 = 1, a, b \in \mathbb{Z} \), has a structure of an abelian group. Prove that equation \( a^2 - 5b^2 = 1 \) has infinitely many integer solutions. (Hint: one solution is \( (9, 4) \).)

5. Let \( \mathbb{F}[[x]] \) be the set of all formal power series in variable \( x \) with coefficients in a field \( \mathbb{F} \). Prove that \( \mathbb{F}[[x]] \) is a ring, and that \( a_0 + a_1x + a_2x^2 + \ldots \) is a unit in this ring iff \( a_0 \neq 0 \).

6. Let \( \mathbb{F}_p \) be the finite field with \( p \) elements (\( p \) is prime). Compute
   
   (a) the number of one-dimensional subspaces in \( \mathbb{F}_p^n \)
   
   (b) \( |GL_2(\mathbb{F}_p)| \)