MAT 534: HOMEWORK 8

DUE WED, NOV 14

Throughout this assignment, $\mathbb{F}$ is an arbitrary field.

1. Which of the following rings are fields? integral domains? In each case, find all invertible elements (also called *units*)
   (a) $R = \mathbb{F}[x]$
   (b) $R = \mathbb{Z}[\omega]$, where $\omega \in \mathbb{C}$ is a primitive cubic root of unity.
   (c) $R = \mathbb{R}[A]$ where $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
   (d) $R = \mathbb{R}[A]$ where $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
   (e) $R = \mathbb{Z}/n\mathbb{Z}$

2. For each of the quotient rings below, answer the following questions: is it a field? is it finite? is it isomorphic to any ring of Problem 1?

(a) $\mathbb{Z}[i]/(2)$	(b) $\mathbb{Z}[i]/(i+1)$	(c) $\mathbb{R}[x]/(x-1)^2$
(d) $\mathbb{R}[x]/(x^2+1)$	(e) $\mathbb{Z}[x]/(2,x)$	(f) $\mathbb{R}[x,y]/(xy)$

3. Let $d \in \mathbb{Z}$, $d > 1$ be squarefree (i.e., d is not divisible by a square of any prime number).
   (a) Show that $\mathbb{Q}[\sqrt{d}] = \{a + b\sqrt{d}, a, b \in \mathbb{Q}\}$ is a field.
   (b) Show that $\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d}, a, b \in \mathbb{Q}\}$ is a an integral domain.
   (c) Define "conjugation" $\overline{} : \mathbb{Q}[\sqrt{d}] \to \mathbb{Q}[\sqrt{d}]$ by $\overline{a + b\sqrt{d}} = a - b\sqrt{d}$. Prove that then $\overline{x+y} = \overline{x} + \overline{y}$, $\overline{xy} = \overline{x} \cdot \overline{y}$.
   (d) Show that $u \in \mathbb{Z}[\sqrt{d}]$ is a unit (i.e., has a multiplicative inverse in $\mathbb{Z}[\sqrt{d}]$) iff $u\overline{u} = \pm 1$.

4. Using the previous problem, show that the set of all solutions of the *Pell equation* $a^2 - db^2 = 1$, $a, b \in \mathbb{Z}$, has a structure of an abelian group. Prove that equation $a^2 - 5b^2 = 1$ has infinitely many integer solutions. (Hint: one solution is $(9, 4)$.)

5. Let $\mathbb{F}[[x]]$ be the set of all formal power series in variable x with coefficients in a field $\mathbb{F}$. Prove that $\mathbb{F}[[x]]$ is a ring, and that $a_0 + a_1x + a_2x^2 + \dots$ is a unit in this ring iff $a_0 \neq 0$.

6. Let $\mathbb{F}_p$ be the finite field with p elements (p is prime). Compute
   (a) the number of one-dimensional subspaces in $\mathbb{F}_p^n$
   (b) $|GL_2(\mathbb{F}_p)|$