MAT 534: HOMEWORK 7

DUE WED, NOV 7

1. Let B be a symmetric bilinear form in a real vector space V. Define a linear operator $i_B\colon V \to V^*$ by $i_B(v) = B(v, -)$ (that is, $i_B(v)$ is the linear functional whose value on a vector $w \in V$ is given by $B(v, w)$). Prove that i_b is an isomorphism iff B is non-degenerate.

2. Consider the (infinite-dimensional) vector space of complex C^∞ functions on $\mathbb{R}$ with compact support. Define the inner product in this space by

$$(f, g) = \int_{\mathbb{R}} \overline{f(x)} g(x)\, dx$$

Is the operator $\frac{d}{dx}$ Hermitian? skew-Hermitian? neither?

3. Find an orthonormal eigenbasis for the operator $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ (in the standard basis of $\mathbb{R}^2$)

4. Let V be a finite-dimensional complex vector space, and $L\colon V \to V$ a diagonalizable operator. Let λ_i be distinct eigenvalues of L and $V_{\lambda_i} = \operatorname{Ker}(L - \lambda_i)$ the corresponding eigenspace, so that $V = \bigoplus V_{\lambda_i}$.

(a) Show that there exist polynomials $p_i \in \mathbb{C}[x]$ such that $p_i(L) = \mathrm{id}$ on V_{λ_i} and $p_i(L) = 0$ on V_{λ_j}.

(b) Show that if $W \subset V$ is stable under action of L (that is, $LW \subset W$), then $W = \bigoplus (W \cap V_{\lambda_i})$.

(c) Show that there exists a subspace W' such that $V = W \oplus W'$ and $LW' \subset W'$.

5. Let V be a finite-dimensional Hermitian space. An operator $L\colon V \to V$ is called *normal* if L, L^* commute.

(a) Show that if an operator L is diagonal in some orthonormal basis, then it is normal.

(b) Show that if L is normal, then L, L^* have a common eigenvector (hint: see problem 5 from the previous homework)

(c) Show that if L is normal, then there is an orthonormal eigenbasis in which it is diagonal.

6. Let B be a symmetric bilinear form in a finite-dimensional real vector space V.

(a) Show that then one can write $V = V_+ \oplus V_0 \oplus V_-$, where subspaces $V_\pm, V_0$ are orthogonal with respect to B (i.e., $B(v_1, v_2) = 0$ if v_1, v_2 are from different subspaces), and restriction of B to V_+ is positive definite, to V_- negative definite, and to V_0 — zero. (Hint: choose some inner product in V, write $B(v, w) = (Av, w)$ for some symmetric operator A, and then diagonalize A.)

(b) Show that if $V = V_+ \oplus V_0 \oplus V_- = V'_+ \oplus V'_0 \oplus V'_-$ are two such decompositions, then $\dim V_+ = \dim V'_+$, $\dim V_- = \dim V'_-$, $\dim V_0 = \dim V'_0$. (Hint: prove that $V'_+ \cap (V_0 \oplus V_-) = \{0\}$.)

(c) Deduce that there exists a basis in which B is diagonal, with $+1, -1$, and 0 on the diagonal, and the number of pluses, minuses, and zeros does not depend on the choice of such a basis. (This is called the *Inertia Theorem*)