# MAT 534: HOMEWORK 5

DUE WED, OCT. 24

Problems marked by asterisk (*) are optional.

**1.** Let $L \subset \mathbb{Z}^n$ be the subgroup generated by rows of an $n \times n$ matrix $A$ with integer entries. Show that if $\det A = 0$, then $\mathbb{Z}^n/L$ is infinite, and if $\det A \neq 0$, then $|\mathbb{Z}^n/L| = |\det A|$.

**2.** Let $V' \subset V$ be a subspace.

(a) Show that there is a canonical isomorphism

$$(V/V')^* = \{f \in V^* \mid f(w) = 0 \ \forall w \in V'\}$$

thus, $(V/V')^*$ is naturally a subspace (not a quotient!) of $V^*$.

(b) More generally, show that for any vector space $W$, one has $\mathrm{Hom}(V/V', W) = \{f \in \mathrm{Hom}(V, W) \mid f(w) = 0 \ \forall w \in V'\}$.

**3.** Let $B\colon V \times V \to \mathbb{K}$ be a symmetric bilinear form

(a) Define, for $v \in V$, $f_v \in V^*$ by $f_v(w) = B(v, w)$. Show that $v \mapsto f_v$ is a linear map.

(b) Assume that $B$ non-degenerate: if $B(v, w) = 0$ for all $w$, then $v = 0$. Prove that then the map $V \to V^*$ constructed in the previous part is an isomorphism.

**4.** Let $T$ be a linear operator on the finite-dimensional space $V$. Suppose there is a linear operator $U$ on $V$ such that $TU = I$. Prove that $T$ is invertible, i.e. has both left and right inverse, and $U = T^{-1}$. Show that this is false when $V$ is not finite-dimensional. (Hint: Let $T = D$ be the differentiation operator on the space of polynomials.)

**5.** Let $U$ and $V$ be subspaces of the same vector space $W$. Verify that $U \cap V$ and $U + V = \{u + v \mid u \in U; v \in V\}$ are also subspaces. Prove that

$$\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)$$

**6.** Let $A\colon V \to V$ be a linear operator on a finite-dimensional space such that $A^2 = A$. Prove that then one can write $V = V_1 \oplus V_2$ so that $A|_{V_1} = \mathrm{id}$, $A_{V_2} = 0$, so $A$ is the projection operator. (Hint: take $V_1 = \mathrm{Im}\, A$, $V_2 = \mathrm{Ker}\, A$.)