MAT 534: HOMEWORK 4

DUE WED, OCT. 10

Problems marked by asterisk (*) are optional.

1. Classify all groups of order 75.

2. Classify all groups of order 20.

3. Prove that $\mathrm{Aut}(\mathbb{Z}_8) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, and use it to describe all semidirect products $\mathbb{Z}_2 \ltimes \mathbb{Z}_8$. (One of them is the dihedral group — which one?)

4. Show that the groups S_3, S_4 are solvable.

5. Let Q be the subgroup in $\mathbb{R}^n$ generated by elements of the form $e_i - e_j$, $i \neq j$, and let $P = \{\lambda \in \mathbb{R}^n \mid \sum \lambda_i = 0, \lambda \cdot \alpha \in \mathbb{Z}\ \forall \alpha \in Q\}$. (Here e_i are the standard generators of $\mathbb{Z}^n$: $e_i = (0, \dots, 1, \dots 0)$, with 1 in the i^{th} place, and $\lambda \cdot \alpha = \sum \alpha_i \lambda_i$ is the usual dot product in $\mathbb{R}^n$.)

Show that P, Q are free abelian groups of rank $n-1$. Show that $Q \subset P$ and describe the quotient P/Q.

6. Let G be the abelian group generated by three elements x, y, z with defining relations $x^3 = xy^2z^3 = 1$. Write this group in the canonical form (i.e., as a product of cyclic groups).

7. Let G be a finitely generated abelian group. By the classification theorem, it can be written as a direct product of a finite group and a free one: there exist subgroups $K, F \leq G$ such that K is finite, F is free (i.e. $F \simeq \mathbb{Z}^r$) and $G = K \times F$. Show that K is defined canonically but F is not: if K', F' are two other subgroups with the same properties (K' is finite, F' is free and $G = K' \times F'$), then $K = K'$ but it is possible that $F \neq F'$. (Of course, F must be isomorphic to F', but they may be different as subgroups of G.)