MAT 534: HOMEWORK 3

DUE WED, OCT. 3

Problems marked by asterisk (*) are optional.

1. Show that if G is a group, and $\varphi_g\colon G \to G$ is conjugation by g: $\varphi_g(x) = gxg^{-1}$, then for any automorphism σ we have $\sigma \circ \varphi_g \circ \sigma^{-1} = \varphi_{\sigma(g)}$. Deduce from this that the group $\mathrm{Inn}(G)$ of inner automorphisms is a normal subgroup in $\mathrm{Aut}(G)$.

2. Show that if $G/Z(G)$ is cyclic, then G is Abelian. (Here $Z(G)$ is the center of G.)

3. (a) Let p be a prime number. Classify all groups of order p.
(b) Classify all groups of order 6.
(c) Let p and q be different prime numbers. Classify all Abelian groups of order pq.
*(d) Let p and q be different prime numbers. Classify all groups of order pq.

4. Prove Cauchy's Theorem for Abelian groups: If G is a finite Abelian group and p is a prime dividing $|G|$, then G contains an element of order p.

Hint: Use complete induction on $|G|$. Namely, show that if the theorem is valid either for a subgroup $H \leq G$ or for the quotient group G/H, then it is valid for G.

5. Let p and q be primes (not necessarily distinct) with $p \leq q$. Prove that if p does not divide $q-1$, then any group of order pq is Abelian.

Hint: Using the class equation, prove that any noncommutative group G of order pq has an element of order q. This element generate the normal cyclic subgroup H of order q. Study the action of G on H by conjugations and compare the resulting automorhisms of H with the possible automorphisms of a cyclic group of order q.

6. Describe all Sylow 2-subgroups and 3-subgroups of D_{12} (symmetries of a regular hexagon).

7. Prove that if $|G| = 105$, then G has a normal Sylow 5-subgroup and a normal Sylow 7-subgroup.