MAT 534: HOMEWORK 2

DUE WED, SEPT. 26

Problems marked by asterisk (*) are optional.

1. Prove that a subgroup of index 2 is always normal.

2. Let D_{2n} be the dihedral group (symmetries of regular n-gon); let $r \in D_{2n}$ be the rotation by angle $2\pi/n$. Let k be a divisor of n and $H = \langle r^k \rangle$ be the subgroup generated by r^k. Show that H is normal in D_{2n} and that $D_{2n}/H \simeq D_{2k}$.

3. (a) Find the sign of the following permutation from S_9:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 8 & 5 & 9 & 7 & 4 & 1 & 2 & 6 \end{pmatrix}$$

(b) Prove that the alternating group A_n is a normal subgroup in S_n.

(c) Prove that A_n is generated by cycles of length 3.

4. (a) Describe all conjugacy classes in S_5. How many elements are in each conjugacy class?

(b) Describe all conjugacy classes in A_5. How many elements are in each conjugacy class?

5. (a) Let H be a normal subgroup in G; denote by $\varphi\colon G \to G/H$ the canonical projection. Prove that for every subgroup $\bar{K} \le G/H$, the preimage $K = \varphi^{-1}(\bar{K}) \subset G$ is a subgroup. Show that if $\bar{K}$ is normal in G/H, then K is normal in G, and $G/K \simeq (G/H)/\bar{K}$.

(b) Prove part (1) of Jordan-Hölder theorem (p. 103 in the textbook).

6. Prove that the Klein group $V_4 = \{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ is a normal subgroup of S_4. Show that S_4/V_4 is isomorphic to S_3