MAT 534: HOMEWORK 11

DUE WED, DEC 12

Throughout this assignment, $\mathbb{F}$ is an arbitrary field.

1. Prove the second isomorphism theorem: if M is a module over a ring R, and $A, B \subset M$ are submodules, then $(A+B)/B = A/(A \cap B)$.

2. (a) Show that if M is an R-module and N — a submodule such that $N, M/N$ are finitely generated, then M is also finitley generated.
(b) Let M be a finitely generated module over a Noetherian ring. Prove that any submodule of M is also finitely generated. [Hint: prove this first for free modules using induction in rank.]

3. (a) Prove that the only ideals in the ring of formal power series $R = \mathbb{F}[[x]]$ are $I_n = (x^n), n \geq 0$. [Hint: remember that $a_0 + a_1 x + \cdots \in R$ is invertible iff $a_0 \neq 0$.]
(b) Prove that a finitely generated $\mathbb{C}[[x]]$-module of zero free rank is the same as a finite-dimensional complex vector space with a nilpotent linear operator $A\colon V \to V$.

4. Dummit and Foote, p. 344, exercise 8.

5. Let M be the module over $\mathbb{C}[A]$ which is two-dimensional as a complex vector space with the action of A given (in some basis) by the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$. Describe the ring $\mathrm{Hom}_{\mathbb{C}[A]}(M, M)$.

6. Dummit and Foote, p. 357, exercise 18.

7. Dummit and Foote, p. 468, exercise 2.

8. Dummit and Foote, p. 469, exercise 11.