1. Prove that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain.

2. Determine the greatest common divisor in $\mathbb{Q}[x]$ of $a(x) = x^3 + 4x^2 + x - 6$ and $b(x) = x^5 - 6x + 5$ and write it as a linear combination of $a(x)$ and $b(x)$.

3. (a) Prove that every $a \in \mathbb{Z}$ can be uniquely written in the form

   $$a = \pm p_1^{n_1} \cdots p_k^{n_k} (q_1 q')^{m_1} \cdots (q_l q')^{m_l}$$

   where $p_i \in \mathbb{Z}$ are integers which are prime (=irreducible) as elements of $\mathbb{Z}[i]$, and $q_i \in \mathbb{Z}[i]$ are irreducible elements of $\mathbb{Z}[i]$ which are not in $\mathbb{Z}$.

   (b) Prove that a prime number $p \in \mathbb{Z}_+$ remains irreducible in $\mathbb{Z}[i]$ iff equation $a^2 + b^2 = p$ has no integer solutions. (Hint: $a^2 + b^2 = (a + bi)(a - bi)$.) Deduce from this that prime numbers of the form $4k + 3$ remain irreducible in $\mathbb{Z}[i]$. (In fact, it is known that a prime integer number is irreducible in $\mathbb{Z}[i]$ iff it has the form $4k + 3$.)

   (c) Assuming the statement given in the previous part, prove that for a positive integer $n$ the following statements are equivalent:

   • $n$ can be written as sum of two squares of integer numbers
   • $n$ can be written in the form $n = z\bar{z}$, $z \in \mathbb{Z}[i]$.
   • In the prime factorization for $n$ (in $\mathbb{Z}$), each prime factor of the form $4k + 3$ has even exponent.

4. Consider the ring $R = \mathbb{Z}[\sqrt{-5}]$.

   (a) Prove that elements $2, 3, 1 \pm \sqrt{-5}$ are irreducible in $R$. [Hint: if $2 = zw$, then $N(z)N(w) = N(2) = 4$, where $N(z) = z\bar{z} \in \mathbb{Z}_+$.]

   (b) Show that $R$ is not UFD because $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$.

   (c) Define the ideals

   $$I = (2, 1 + \sqrt{-5})$$
   $$J = (3, 2 + \sqrt{-5})$$
   $$J' = (3, 2 - \sqrt{-5})$$

   Prove that these ideals are prime (see hint in in Exercise 8, p. 293 in the book).

   (d) Prove that $(2) = I^2$, $(3) = JJ'$, $(1 - \sqrt{-5}) = IJ$, $(1 + \sqrt{-5}) = IJ'$. Deduce from this that both factorizations $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ give the same presentation for $(6)$ as a product of prime ideals: $(6) = I^2JJ'$.

5. In each of the following cases, determine whether $a$ is an irreducible element of the ring $R$. Is $a$ a prime?

   (a) $R = \mathbb{F}_2[x], a = x^2 + x + 1$
   (b) $R = \mathbb{Z}[\sqrt{5}], a = 2$
   (c) $R = \mathbb{Q}[x], a = x^4 + x^3 + x^2 + x + 1$
   (d) $R = \mathbb{Q}[x], a = x^4 + 4$