MAT 534: HOMEWORK 1

DUE WED, SEPT. 19

Problems marked by asterisk (*) are optional.

1. Let G be a group. Prove that if $x^2 = 1$ for every $x \in G$, then G is abelian.

2. For each pair of positive integers m and n find all homomorphisms from $\mathbb{Z}_n$ to $\mathbb{Z}_m$.

3. Prove that any subgroup in $\mathbb{Z}$ must be of the form $H = a \cdot \mathbb{Z}$ for some $a \in \mathbb{Z}$.

4. Find the cycle decompositon of the following permutation from S_9:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 8 & 5 & 9 & 7 & 4 & 1 & 2 & 6 \end{pmatrix}$$

What is the order of σ?

5. (a) Prove that the group of symmetries of an equilateral triangle is isomorphic to S_3.
(b) Prove that the group of rotations of a cube is isomorphic to S_4. Describe the stabilizer of a vertex; of an edge.
(c) For each pair of parallel faces of a cube consider the line passing through the centers of the faces. Using that all rotations of a cube permute these lines, construct an epimorphism $S_4 \to S_3$.

6. Let $H \le K \le G$ be subgroups. Prove that then $|G : H| = |G : K| \cdot |K : H|$.

7. Let p be a prime number and $\mathbb{Z}_p^\times$— the group of all non-zero remainders modulo p (with respect to multiplication). Deduce from Lagrange theorem that for any integer a not divisible by p, we have $a^{p-1} \equiv 1 \mod p$.

***8.** How many ways are there to group numbers $\{1 \dots 2n\}$ into pairs? Order of pairs and order inside each pair is not important. For example, for $n = 2$, there are three ways:

(12) (34)

(13) (24)

(14) (23)

(Hint: first show that one can define a transitive action of S_{2n} on the set of all such pairings.)