MAT 534: HOMEWORK 8

DUE THURSDAY, NOV 17

Throughout this assignment, R is an arbitrary ring with unit (not necessarily commutative), and $\mathbb{F}$ is a field.

1. Let $R = \mathrm{Mat}_{n\times n}(\mathbb{F})$ be the ring of $n \times n$ matrices with coefficients in $\mathbb{F}$. Let $M = \mathbb{F}^n$ (column vectors), considered as a left module over R.

Consider R as left module over itself. Prove that then $R \simeq M \oplus M \oplus \cdots \oplus M$ (direct sum of n copies of M).

2. (a) Show that if M is an R-module and N — a submodule such that $N, M/N$ are finitely generated, then M is also finitely generated.

(b) Recall that a commutative ring R is called Noetherian if every ideal is finitely generated. Show that for such a ring, any submodule in R^n is finitely generated. [Hint: consider morphism of R-modules $R^n \to R$: $(a_1, \dots, a_n) \mapsto a_1$.]

(c) Show that if M is a finitely generated module over a Noetherian ring, then any submodule of M is also finitely generated.

3. A module M over a (not necessarily commutative) unital ring R is called *simple* if it has no nonzero proper submodules.

(a) Prove that every simple module is generated by a single element.

(b) Prove that every simple module is isomorphic to a module of the form R/I, where $I \subset R$ is a maximal left ideal.

(c) Describe all simple modules over $\mathbb{C}[x]$; over $\mathbb{R}[x]$.

4. Dummit and Foote, p. 344, exercise 8.

5. Dummit and Foote, p. 344, exercise 9.

6. Let M be a module over a PID R and $a \in R$ annihilates M: $am = 0$ for any $m \in M$. Assume that $a = a_1 \dots a_n$, where a_i are pairwise relatively prime. Prove that then

$$M = M_1 \oplus \cdots \oplus M_n, \qquad M_i = \{m \in M \mid a_i M = 0\}$$

[Hint: first prove it for $n = 2$ and then use induction.]

7. Dummit and Foote, p. 469, exercise 11.

8. Dummit and Foote, p. 469, exercise 12.