MAT 534: FINAL EXAM

TH, DEC 14, 2023

Your name:____

(please print)

	1	2	3	4	5	**Total**
Grade						

1. Describe all semidirect products $\mathbb{Z}_4 \ltimes \mathbb{Z}_{11}$, writing each of of them by generators and relations. Which of them are isomorphic?

2. Let G be a finite group and let $p \in \mathbb{Z}$ be the smallest divisor (greater than 1) of $n = |G|$. Let $H \subset G$ be a subgroup of index p. Consider the coset space $X = G/H$ and let H act on this space by multiplication on the left: $h \cdot (xH) = hxH$.

(a) Show that for any orbit S of this action, either $|S| = 1$ or $|S| \geq p$.

(b) Show that every orbit consists of a single element. Deduce from this that H is normal in G.

3. Consider the ring $R = \mathbb{F}_{17}[x]/(x^3 - 3x^2 + 4)$, where $\mathbb{F}_{17} = \mathbb{Z}/(17)$ is the field of 17 elements.

(a) Describe all ideals in R. Which of them are maximal? Is it true that R is a direct sum of fields?

(b) Find R-modules M_1, M_2, M_3 such that any finitely generated module over R is isomorphic to direct sum of copies of M_1, M_2, M_3.

4. A module M over a ring R is called *Noetherian* if every increasing chain of submodules $M_1 \subset M_2 \subset \cdots \subset M$ stabilizes after finitely many steps. (This is very similar to the definition of a Noetherian ring – but here we are talking about modules over arbitrary rings).

(a) Prove that any PID, considered as a module over itself, is Noetherian.

(b) Prove that if M is a Noetherian module, and $f\colon M \to M$ is surjective, then it is also injective.

5. Let V be a finite-dimensional vector space over $\mathbb{C}$. Recall that a linear operator $L\colon V \to V$ is called nilpotent if $L^k = 0$ for some k.

Show that any operator $A\colon V \to V$ can be written as a sum $A = A_s + A_n$, where A_s is diagonalizable, A_n is nilpotent, and A_s, A_n commute. Moreover, show that if B is another operator which commutes with A, then B also commutes with A_s, A_n.