MAT 534: HOMEWORK 8  
DUE MON, NOV 29

Throughout this assignment, \( \mathbb{F} \) is an arbitrary field.

1. Which of the following rings are fields? integral domains? In each case, find all invertible elements (also called units)
   (a) \( R = \mathbb{F}[x] \)
   (b) \( R = \mathbb{Z}[\omega] \), where \( \omega \in \mathbb{C} \) is a primitive cubic root of unity.
   (c) \( R = \mathbb{R}[A] \) where \( A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \)
   (d) \( R = \mathbb{R}[A] \) where \( A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \)
   (e) \( R = \mathbb{Z}/n\mathbb{Z} \)

2. Let \( d \in \mathbb{Z}, \ d > 1 \) be squarefree (i.e., \( d \) is not divisible by a square of any prime number).
   (a) Show that \( \mathbb{Q}[\sqrt{d}] = \{a + b\sqrt{d}, a, b \in \mathbb{Q}\} \) is a field.
   (b) Show that \( \mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d}, a, b \in \mathbb{Q}\} \) is an integral domain.
   (c) Define “conjugation” \( \overline{\sqrt{d}} : \mathbb{Q}[\sqrt{d}] \to \mathbb{Q}[\sqrt{d}] \) by \( a + b\sqrt{d} = a - b\sqrt{d} \). Prove that then \( \overline{x + y} = \overline{x} + \overline{y}, \overline{xy} = \overline{x} \cdot \overline{y} \).
   (d) Show that \( u \in \mathbb{Z}[\sqrt{d}] \) is a unit (i.e., has a multiplicative inverse in \( \mathbb{Z}[\sqrt{d}] \)) iff \( u\overline{u} = \pm 1 \).

3. Using the previous problem, show that the set of all solutions of the Pell equation \( a^2 - db^2 = 1, a, b \in \mathbb{Z} \), has a structure of an abelian group. Prove that equation \( a^2 - 5b^2 = 1 \) has infinitely many integer solutions. (Hint: one solution is (9, 4).)

4. Let \( \mathbb{F}[[x]] \) be the set of all formal power series in variable \( x \) with coefficients in a field \( \mathbb{F} \). Prove that \( \mathbb{F}[[x]] \) is a ring, and that \( a_0 + a_1x + a_2x^2 + \ldots \) is a unit in this ring iff \( a_0 \neq 0 \).

5. Let \( \mathbb{F}_p \) be the finite field with \( p \) elements (\( p \) is prime). Compute
   (a) the number of one-dimensional subspaces in \( \mathbb{F}_p^n \)
   (b) \( |GL_2(\mathbb{F}_p)| \)