Problems marked by asterisk (*) are optional.

For some problems might need the following basic result from number theory (we will prove it later): an integer \( k \) is invertible modulo \( n \) if and only if \( k, n \) are relatively prime.

1. Do the problem discussed in class: if \( x \in G \) is an element of order \( n \), then the subgroup generated by \( x \) is isomorphic to \( \mathbb{Z}_n \).

2. Construct a bijection between the coset space \( S_n/S_k \times S_{n-k} \) and the set \( B \) of all sequences of \( k \) zeroes and \( n-k \) ones. (The map was discussed in class; you need to show that it is a bijection).

3. Prove that any subgroup of index 2 is normal.

4. Describe all subgroups of symmetric group \( S_3 \). For each of them, say whether it is normal; if it is, describe the quotient.

5. Prove that any subgroup in \( \mathbb{Z} \) must be of the form \( H = a \cdot \mathbb{Z} \) for some \( a \in \mathbb{Z} \).

6. Let \( p \) be a prime number and \( \mathbb{Z}_p^x \)— the group of all non-zero remainders modulo \( p \) (with respect to multiplication). Deduce from Lagrange theorem that for any integer \( a \) not divisible by \( p \), we have \( a^{p-1} \equiv 1 \mod p \).

7. (a) Prove that an element \( k \in \mathbb{Z}_n \) is a generator of \( \mathbb{Z}_n \) if and only if \( k \) is relatively prime with \( n \).

(b) A complex number \( \zeta \) is called a primitive root of unity of order \( n \) if \( \zeta^n = 1 \), but for all \( k = 1, 2, \ldots n-1 \), we have \( \zeta^k \neq 1 \). How many primitive roots of unity of order 15 are there? Describe them all.