Solutions of practice midterm:

1. \[ A \Rightarrow (B \land (C \lor D)) \]
   (It can also be argued that correct interpretation is
   \[ A \leftrightarrow (B \land (C \lor D)) \])

2. 
   a. True. Take \( x = 0 \); then for any \( y \), \( xy = 0 < 1 \)
   b. False. Take \( x = 0 \); then there is no \( y \) which
      would give \( xy = 1 \).
   c. True. Let \( x > 0, y > 0 \). Then \( y/x > 0 \).
      Choose any positive integer \( n \) such that
      \( n > y/x \); then \( nx > y \).

   (We are using the Archimedean property of real numbers: for any positive real
   number \( t \), there exists an integer \( n > t \).
   This requires a proof; such a proof can
   only be given using completeness axiom of \( \mathbb{R} \) and is usually done in analysis
   class).
(3) Proof by induction:

- **Base case:** \( n=2 \).

\[
\frac{1}{3} + \frac{1}{4} = \frac{7}{12} > \frac{13}{24}
\]

- **Induction step**

Assume \( a_n > \frac{13}{24} \).

We need to prove that then, \( a_{n+1} > \frac{13}{24} \).

where \( a_n = \frac{1}{n+1} + \cdots + \frac{1}{2n} \).

Indeed:

\[
a_{n+1} = \frac{1}{n+2} + \cdots + \frac{1}{2n+1} + \frac{1}{2n+2}
\]

\[
= a_n - \frac{1}{n+1} + \frac{1}{2n+1} + \frac{1}{2n+2}
\]

\[
= a_n + \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{2}{2n+2}
\]

\[
= a_n + \frac{1}{2n+1} - \frac{1}{2n+2}
\]

\[
= a_n + \frac{1}{(2n+1)(2n+2)} > a_n.
\]

Thus, \( a_{n+1} > a_n \), so if \( a_n > \frac{13}{24} \), then

\[
a_{n+1} > \frac{13}{24}
\]
Yes, it is an equivalence relation.

- Reflexive: \((x, y) \sim (x, y)\)
  \[x + y = x + y\]  - true

- Symmetric: \((x_1, y_1) \sim (x_2, y_2) \iff (x_2, y_2) \sim (x_1, y_1)\)
  \[
  (x_1 + y_1 = x_2 + y_2) \iff (x_2 + y_2 = x_1 + y_1)
  \]  true

- Transitive:
  \[
  (x_1, y_1) \sim (x_2, y_2) \sim (x_3, y_3) \implies (x_1, y_1) \sim (x_3, y_3)
  \]
  \[
  x_1 + y_1 = x_2 + y_2 \quad \text{and} \quad x_2 + y_2 = x_3 + y_3 
  \]  \[
  \implies x_1 + y_1 = x_3 + y_3
  \]
  true by transitivity of equality

\[
[(1, 1)] = \{(x, y) \mid x + y = 2\}
\]

No. It is not transitive:

\[
(0, 0) \not\sim (0, 1)
(0, 1) \not\sim (1, 1)
\]

but \((0, 0)\) is not equivalent to \((1, 1)\).
Left hand side:
\[ A - UB_i = \{x \mid x \notin A \land x \notin UB_i \} \]
\[ = \{x \mid x \in A \land (\forall i: x \notin B_i) \} \]
Thus, \[ x \in A - UB_i \iff x \in A \land (\forall i: x \notin B_i) \] (1)

RHS:
\[ x \in \bigcap (A - B_i) \iff (\forall i: x \in A - B_i) \]
\[ \iff (\forall i: x \in A \land x \notin B_i) \] (2)
Thus, if \( x \in A - UB_i \), then \( x \in A \) and \( \forall i: x \notin B_i \), then \( \exists i \): for any \( i \), \( (x \in A \land x \notin B_i) \) is true.
Conversely, if \( x \in \bigcap (A - B_i) \), then
\[ \forall i: (x \in A \land x \notin B_i) \]
so \( x \in A \), and for any \( i \), \( x \notin B_i \), so
\[ x \in A \land (\forall i: (x \notin B_i)) \]
Thus, \( (x \in A - UB_i) \iff (x \in \bigcap (A - B_i)) \)
6. It is bijective; any \((y_1, y_2) \in \mathbb{R}^2\) has a unique preimage:

\[
\begin{cases}
  x_1 - x_2 = y_1 \\
  3x_2 = y_2
\end{cases} \iff \begin{cases}
  x_1 = y_1 + x_2 = y_1 + \frac{1}{3} y_2 \\
  x_2 = \frac{1}{3} y_2
\end{cases}
\]

The inverse function is

\[
f^{-1}(y_1, y_2) = (y_1 + \frac{1}{3} y_2, \frac{1}{3} y_2)
\]

6. As in 5, it is injective, but not surjective: \((1, 1)\) has no preimages since equation \(3x_2 = 1\) has no solutions with integer \(x_2\).

7. a. Let \(A = \mathbb{Z} \times \mathbb{Z}\) be the set of all points with integer coordinates. Let \(T\) be the set of all integer triangles. For any triangle, choose an ordering of its vertices. This gives, for any triangle \(t\), a triple of points \((a_1, a_2, a_3) \in A \times A \times A\). This gives injection \(T \rightarrow A \times A \times A\).

Since cartesian product of \(A\) is denumerable
sets is denumerable, $A$ is denumerable, and so is $A \times A \times A$. Since any subset of a denumerable set is finite or denumerable, $T$ is denumerable.

\[b\] Set of all triangles contains as subset set $S$ of triangles with vertices

\[
\begin{align*}
(0,0) \\
(0,1) \\
(1,1) \\
S & \subset \mathbb{R}
\end{align*}
\]

Thus, $S$ is in bijection with $\mathbb{R}$, so it is not denumerable. Thus, any set which has $S$ as a subset is not denumerable either.